

# Partial differential equations of non-linear Schrödinger type in Hölder spaces: non-degeneracy and multibumps

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## Abstract

It is shown that under suitable conditions a solution of the equation  $-\Delta u + F(x, u) = 0$  that is non-degenerate for Sobolev spaces is also non-degenerate for Hölder spaces. This leads to the construction of uncountably many geometrically distinct solutions with infinitely many bumps for the equation  $-\Delta u + F(V(\varepsilon x), u) = 0$  for small  $\varepsilon$ .

## 1 Introduction

We give here a brief outline of the contents and background of the present paper leaving precise definitions and references to later sections.

In this paper we build on a result of a previous paper which examined the non-degeneracy with respect to the Sobolev spaces  $W^{2,2}(\mathbb{R}^n)$  and  $L^2(\mathbb{R}^n)$  of certain non-trivial solutions of the equation

$$-\Delta u + F(V(\varepsilon x), u) = 0 \tag{1.1}$$

for small  $\varepsilon$ . We now seek to draw similar conclusions for (1.1) with respect to the Hölder spaces  $C^{2,\lambda}(\mathbb{R}^n)$  and  $C^{0,\lambda}(\mathbb{R}^n)$ . This is the content of sections 2 and 3.

Equation (1.1) is a general version of the problem of semi-classical states for the non-linear Schrödinger equation, usually given in the form

$$-\Delta u + V(\varepsilon x)u - u^p = 0$$

where  $\varepsilon$  is a small parameter, so that the spatial function  $V(\varepsilon x)$  is nearly constant.

It is known that non-degenerate solutions can be used to construct multibump solutions. Roughly speaking, a multibump solution is one that is near to a sum of given solutions which are approximately concentrated on disjoint sets. The result of section 3, giving non-degeneracy in Hölder spaces, then opens the way in section 4 to the construction of multibump solutions of (1.1) for sufficiently small  $\varepsilon$ . Unlike for the case of Sobolev spaces, the approximation in Hölder space norms allows for infinitely many bumps.

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\*The second author was supported by a grant from the University of Iceland Eimskip Fund

## 2 Two notions of non-degeneracy

We first recall the definition of the Hölder space  $C^{m,\lambda}(\mathbb{R}^n)$ , where  $m$  is an integer and  $0 < \lambda < 1$ . It consists of  $C^m$  functions  $u$  on  $\mathbb{R}^n$ , whose derivatives up to order  $m$  are bounded, and such that there exists  $C > 0$  with

$$|D^\beta u(x) - D^\beta u(y)| \leq C|x - y|^\lambda$$

for every  $x, y \in \mathbb{R}^n$ , and every  $\beta$  with  $|\beta| = m$ . It is endowed with the norm

$$\|u\|_{C^{m,\lambda}} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_\infty + \sup_{x,y \in \mathbb{R}^n, |\beta|=m} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^\lambda}$$

which turns  $C^{m,\lambda}(\mathbb{R}^n)$  into a Banach space. These spaces are very useful when studying elliptic equations, since they are the natural spaces of classical solutions on which elliptic regularity theorems may be applied.

In the following, we write  $C^{m,\lambda}$  for  $C^{m,\lambda}(\mathbb{R}^n)$ . We also write  $H^2$  for the Sobolev space  $W^{2,2}(\mathbb{R}^n)$  and  $L^2$  for  $L^2(\mathbb{R}^n)$ . We shall sometimes need the space  $C^m(\mathbb{R}^n)$ , meaning the Banach space of  $C^m$  functions whose derivatives up to order  $m$  are bounded over all of  $\mathbb{R}^n$ . At one point (in a later section) we use the notation  $C^{m,\lambda}(\overline{V})$  where  $V$  is an open set in  $\mathbb{R}^n$ . This refers to the Banach space of functions whose derivatives up to order  $m$  are bounded in  $V$ , and whose derivatives of order  $m$  possess a finite Hölder semi-norm with exponent  $\lambda$  on  $V$ .

We consider the problem

$$-\Delta u + F(x, u) = 0$$

on  $\mathbb{R}^n$ , where the non-linearity  $F$  is a  $C^3$  function. We say that a solution  $U$  is *non-degenerate for Sobolev spaces* if the linearized operator

$$-\Delta + D_u F(x, U(x))$$

is invertible from  $H^2$  to  $L^2$ . We say that  $U$  is *non-degenerate for Hölder spaces* if the linearized operator is invertible from  $C^{2,\lambda}$  to  $C^{0,\lambda}$ . Our object in this section is to show that under suitable conditions a solution  $U$  that lies both in  $H^2$  and  $C^{2,\lambda}$ , and is non-degenerate for Sobolev spaces, is also non-degenerate for Hölder spaces.

We set out growth conditions on  $F$ , which fall into two classes, and a positivity condition:

### (G1) Growth conditions

$$\begin{aligned} |F(x, u)| &\leq C(|u| + |u|^{\alpha_1}) \\ |D_u F(x, u)| &\leq C(1 + |u|^{\alpha_2}) \end{aligned}$$

where

$$1 \leq \alpha_1 \leq \frac{n}{n-4}, 0 \leq \alpha_2 \leq \frac{4}{n-4}$$

with no upper bound if  $n \leq 4$ .

These growth conditions imply that if  $U \in H^2$ , then the function  $F(x, U(x))$  will be in  $L^2$ , and that the operator

$$v \mapsto D_u F(x, U(x))v$$

is well-defined from  $H^2$  to  $L^2$ , see [4].

**(G2) Growth conditions**  $D_{xu}F(x, u)$ ,  $D_{uu}F(x, u)$  and  $D_{xuu}F(x, u)$  are bounded uniformly with respect to  $x$  if  $u$  belongs to a bounded subset of  $\mathbb{R}$ .

This in particular ensures that if  $U \in C^{2,\lambda}$  then the function  $F(x, U(x))$  will be in  $C^{0,\lambda}$ , and that the operator

$$v \mapsto D_u F(x, U(x))v$$

is well-defined from  $C^{2,\lambda}$  to  $C^{0,\lambda}$ .

**(P) Positivity condition**

$$0 < h < \liminf_{|x| \rightarrow +\infty} D_u F(x, 0)$$

for some constant  $h$ .

**Theorem 2.1** *Assume the conditions (G1), (G2) and (P). Let  $U \in C^{2,\lambda} \cap H^2$  be a solution of*

$$-\Delta u + F(x, u) = 0$$

*such that  $U$  and  $DU$  tend to 0 at infinity. Assume that  $U$  is a non-degenerate solution in the sense of Sobolev spaces. Then  $U$  is also non-degenerate in the sense of Hölder spaces.*

**Proof.** Let

$$A = -\Delta + D_u F(x, U(x)) : C^{2,\lambda} \rightarrow C^\lambda$$

Introduce the limit operator

$$A_\infty = -\Delta + D_u F(x, 0)$$

also from  $C^{2,\lambda}$  to  $C^{0,\lambda}$ . The plan is the following: show that  $A$  is a compact perturbation of  $A_\infty$ , then show that  $A_\infty$  is Fredholm of index 0 (using our positivity assumption), deduce that  $A$  is Fredholm of index 0 and finally use a result from [1] to prove that  $A$  is one-to-one.

The difference  $A - A_\infty$  is the operator of multiplication by the function

$$h(x) := D_u F(x, U(x)) - D_u F(x, 0)$$

with derivative

$$Dh(x) = D_{xu} F(x, U(x)) - D_{xu} F(x, 0) + D_{uu} F(x, U(x)) DU(x).$$

By the boundedness assumptions on  $D_{uu} F$  and  $D_{xuu} F$ , together with the assumptions that  $U$  and  $DU$  tend to 0 at infinity, we find that  $h$  and  $Dh$  tend to 0 at infinity. To show that  $A - A_\infty$  is a compact operator from  $C^{2,\lambda}$  to  $C^\lambda$  we let  $(w_n)$  be a bounded sequence in  $C^{2,\lambda}$ . Using Ascoli's theorem, we can find a function  $w \in C^1(\mathbb{R}^n)$  and a subsequence of  $w_n$  (still denoted  $w_n$ ) that converges to  $w|_B$  in the norm of  $C^1(B)$  for every bounded, open  $B \subset \mathbb{R}^n$ . But since  $h$  and  $Dh$  vanish at infinity we see that  $hw_n$  converges to  $hw$  in  $C^1(\mathbb{R}^n)$ , and hence also in  $C^{0,\lambda}$ . Therefore  $A$  is a compact perturbation of  $A_\infty$ .

We now show that  $A_\infty$  is Fredholm of index 0. Let  $V$  be a  $C^1$  function on  $\mathbb{R}^n$  such that

- $V(x) \geq \delta > 0$  pointwise (for some  $\delta$ ).
- $V(x) = D_u F(x, 0)$  for  $|x| > M$  for some  $M$ .
- $V$  and  $DV$  are globally bounded.

Such a function can be obtained by taking a smooth, suitably close uniform approximation to the function  $\max\{h/2, D_u F(x, 0)\}$  in a large enough ball (with radius  $R$ , say) and pasting it smoothly to  $D_u F(x, 0)$  in the complement of the ball of radius  $R - 1$ .

The argument in the second paragraph shows that the operator of multiplication by  $D_u F(x, 0) - V(x)$  from  $C^{2,\lambda}$  to  $C^{0,\lambda}$  is compact; hence it is sufficient to show that the operator  $T := -\Delta + V(x)$ , from  $C^{2,\lambda}$  to  $C^{0,\lambda}$ , is invertible. This is a consequence of the inequalities  $V(x) \geq \delta > 0$  as we proceed to show.

Let  $u_0 \in C^{2,\lambda}$  such that

$$Tu_0 := -\Delta u_0 + V(x)u_0 = 0.$$

Recall that

$$0 < \delta < V(x) < H$$

for some constants  $\delta$  and  $H$ . Let  $m = (\delta + H)/2$ , and write  $V(x) = m + g(x)$ . By the properties of  $V$ ,  $g \in C^\lambda$  and  $\|g\|_\infty \leq (H - \delta)/2$ .

Let  $E_m$  be the radially symmetric fundamental solution of the Helmholtz operator  $-\Delta + m$  where  $m > 0$ . It is known that  $E_m$  and its first order partial derivatives are in  $L^1$  and that  $\|E_m\|_{L^1} = 1/m$ , and therefore it defines a convolution operator, which, considered from  $L^\infty$  to itself has norm  $1/m$ , and which also maps  $L^\infty$  into  $C^1(\mathbb{R}^n)$ . Moreover the only bounded solution of  $-\Delta u + mu = 0$  when  $m > 0$  is  $u = 0$  (indeed

application of the Fourier transform shows that  $u = 0$  is the only solution that is a tempered distribution). Hence if  $f \in L^\infty$  then  $u = E_m * f$  is the unique bounded solution of  $-\Delta u + mu = f$ . Moreover it is known that if  $f \in C^{0,\lambda}$  then  $E_m * f \in C^{2,\lambda}$ .

The equation satisfied by  $u_0$ , which can be rewritten as  $-\Delta u_0 + mu_0 + g(x)u_0 = 0$ , is therefore equivalent to

$$u_0 + E_m * (gu_0) = 0.$$

But the operator  $v \mapsto E_m * (gv)$  from  $L^\infty$  to itself, has norm less than or equal to

$$\frac{1}{m} \frac{H - \delta}{2} = \frac{H - \delta}{H + \delta} < 1$$

and so the operator  $v \mapsto v + E_m * (gv)$  is invertible from  $L^\infty$  to itself. We therefore conclude that  $u_0 = 0$  so the operator  $T$  is injective.

To show that  $T$  is surjective let  $v \in C^\lambda$ . Then  $E_m * v$  is bounded, and so there is a unique bounded function  $w$  such that

$$w + E_m * (gw) = E_m * v$$

which is equivalent to saying that  $w$  is the unique bounded solution of

$$-\Delta w + mw + g(x)w = v$$

in the sense of distributions. Since  $w = E_m * (v - gw)$  we see that  $w \in C^1(\mathbb{R}^n)$  and hence  $v - gw \in C^{0,\lambda}$ . We conclude that  $w \in C^{2,\lambda}$ . Thus  $T$  is surjective and we conclude that it is invertible.

Since  $A$  is a compact perturbation of  $A_\infty$ , which is itself a compact perturbation of  $T$ , this shows that  $A$  is Fredholm of index 0.

Now it only remains to prove that  $A$  is one-to-one. Let  $u \in C^{2,\lambda}$  be such that

$$-\Delta u + D_u F(x, U(x))u = 0$$

Because of the positivity assumption, the potential  $D_u F(x, U(x))$  satisfies

$$\liminf_{|x| \rightarrow +\infty} D_u F(x, U(x)) > h$$

since  $U$  tends to 0 at infinity. Therefore the quantity

$$\Sigma := \sup_K \inf \left\{ \int (|\nabla \varphi|^2 + D_u F(x, U(x))|\varphi|^2) dx, \varphi \in \mathcal{D}(\mathbb{R}^n \setminus K), \int |\varphi|^2 = 1 \right\},$$

where the supremum is taken over compact subsets  $K$  of  $\mathbb{R}^n$ , satisfies

$$\Sigma \geq h.$$

Now it is known that  $\Sigma$  is the infimum of the essential spectrum of the operator  $-\Delta + D_u F(x, U(x))$  defined from  $H^2$  to  $L^2$ . Because 0 is below the essential spectrum, it is possible to apply [1, theorem 4.1] for functions in the kernel of  $-\Delta + D_u F(x, U(x))$ , which yields

$$\int |u(x)|^2 e^{2\gamma|x|} dx < +\infty$$

for every  $\gamma < \sqrt{\hbar}$ . In particular,  $u \in L^2$ . All growth and regularity assumptions on  $u$  needed in [1] are satisfied since  $u \in C^{2,\lambda}$ .

By elliptic regularity,  $u$  is actually in  $H^2$ , and therefore belongs to the kernel of  $-\Delta + D_u F(x, U(x)) : H^2 \rightarrow L^2$ . So  $u = 0$  almost everywhere, because  $U$  is non-degenerate between Sobolev spaces, and so  $u = 0$  everywhere since it is continuous. This proves that  $A$  is one-to-one, hence invertible.

### 3 The non-linear Schrödinger equation in Hölder spaces

Let us now consider the problem

$$-\Delta u + F(a, u) = 0$$

where  $a$  is a real parameter, and  $F$  satisfies some standard growth conditions that we will describe later. Assume that  $F(a, 0) = 0$  for each  $a$ , so that the zero function is a solution. Assume also that for each  $a$ , we know a solution  $\Phi_a$  possessing some non-degeneracy: the operator

$$-\Delta + D_u F(a, \Phi_a(x))$$

from  $H^2$  to  $L^2$  has its kernel spanned by the  $n$  partial derivatives of  $\Phi_a$ , and the range is the orthogonal complement in  $L^2$  of the kernel. We perturb the equation, by putting a nearly constant function instead of  $a$ :

$$-\Delta u + F(V(\varepsilon x), u) = 0 \tag{3.1}$$

where  $\varepsilon$  is a small positive parameter, and  $V$  is a  $C^3$  function. Problems of this form arise in the study of semiclassical states of the non-linear Schrödinger equation

$$-\Delta u + V(\varepsilon x)u - u^p = 0.$$

In the next four paragraphs we detail our assumptions about  $F$ ,  $\Phi_a$  and  $V$ .

**(S1) Properties of  $F$ .** We assume that  $F$  is a  $C^2$  map, such that the derivatives  $\frac{\partial^3 F}{\partial u^3}$  and  $\frac{\partial^3 F}{\partial^2 u \partial a}$  exist and are continuous. We impose the following growth conditions on  $F$  :

$$\begin{aligned} |F(a, u)|, \left| \frac{\partial F}{\partial a}(a, u) \right|, \left| \frac{\partial^2 F}{\partial a^2}(a, u) \right| &\leq C(|u| + |u|^{\alpha_1}) \\ \left| \frac{\partial F}{\partial u}(a, u) \right|, \left| \frac{\partial^2 F}{\partial u \partial a}(a, u) \right| &\leq C(1 + |u|^{\alpha_2}) \\ \left| \frac{\partial^2 F}{\partial u^2}(a, u) \right|, \left| \frac{\partial^3 F}{\partial u^2 \partial a}(a, u) \right| &\leq C(1 + |u|^{\alpha_3}) \\ \left| \frac{\partial^3 F}{\partial u^3}(a, u) \right| &\leq C(1 + |u|^{\alpha_4}) \end{aligned}$$

for non-negative exponents  $\alpha_i$ , without any upper limits if  $1 \leq n \leq 4$ , whereas for  $n = 5$  we assume

$$1 \leq \alpha_1 < 5, \quad \alpha_2 \leq 2, \quad \alpha_3 < 3, \quad \alpha_4 \leq 2$$

**(S2) Properties of  $\phi_a$ .** The function  $\phi_a(x)$  is a solution to  $-\Delta u + F(a, u) = 0$  in  $H^2$  and has the following properties :

1.  $\phi_a(x) = \Phi_a(r)$  is spherically symmetric.
2.  $\int \frac{\partial F}{\partial a}(a, \Phi_a(r)) \Phi'_a(r) r \, dx \neq 0$ .
3.  $\phi_a$  and its first derivatives have exponential decay.
4.  $\phi_a$  is a quasi-non-degenerate solution, that is the operator

$$-\Delta + \frac{\partial F}{\partial u}(a, \phi_a(x)) : H^2 \rightarrow L^2$$

has as its kernel the space spanned by the  $n$  partial derivatives  $D_j \phi_a(x)$ , which are assumed to be independent, and its range is the space in  $L^2$  orthogonal to its kernel. This implicitly says that the partial derivatives belong to  $H^2$ .

These properties hold for the ground state  $\phi_a$  of the non-linear Schrödinger equation

$$-\Delta u + au - u^p = 0 \tag{3.2}$$

see [3], [7] and [8].

**(S3) Properties of  $V$ .** The function  $V$  is  $C^3$  with its range in the interval  $I$ . It and its first partial derivatives are bounded, while its second and third partial derivatives have polynomial growth.

**(S4) Positivity assumption.** There exists  $\delta > 0$  such that

$$\frac{\partial F}{\partial u}(a, 0) > \delta$$

for all  $a$  in the range of  $V$ .

The following theorem summarizes the main results concerning the problem 3.1 that were proved in [4] and [5].

**Theorem 3.1** *Let  $F$ ,  $\phi_a$  and  $V$  satisfy the conditions (S1), (S2), (S3) and (S4). Let  $b$  be a non-degenerate critical point of  $V$ , and  $a = V(b)$ . Then, for sufficiently small  $\varepsilon > 0$ , the equation  $-\Delta u + F(V(\varepsilon x), u) = 0$  has a solution in  $H^2$  of the form*

$$u_\varepsilon(x) = \phi_a \left( x - \frac{b}{\varepsilon} + s_\varepsilon \right) + \varepsilon^2 w_\varepsilon \left( x - \frac{b}{\varepsilon} + s_\varepsilon \right)$$

where  $s_\varepsilon \in \mathbb{R}^n$ ,  $w_\varepsilon \in H^2$  and is orthogonal to the partial derivatives  $D_j \phi_a$ . Both  $s_\varepsilon$  and  $w_\varepsilon$  depend continuously on  $\varepsilon$ . As  $\varepsilon$  tends to 0,  $s_\varepsilon$  tends to 0 and  $w_\varepsilon$  tends to a computable function  $\eta \in H^2$ . [A prescription is given in [4] for  $\eta$  but is not needed here.]

Furthermore, the operator

$$T_\varepsilon := -\Delta + \frac{\partial F}{\partial u}(V(\varepsilon x), u_\varepsilon)$$

from  $H^2$  to  $L^2$  is invertible and the norm of its inverse is  $O(\varepsilon^{-2})$  as  $\varepsilon \rightarrow 0$ .

Our main objective in this section is to derive some properties of the solutions  $u_\varepsilon$  pertaining to their membership of Hölder spaces. For this purpose we shall need an interpolation lemma, probably well-known, but include a proof for the sake of completeness. It shows how bounds on a function and its second derivative imply a bound on the first derivative. It will be used later on to obtain exponential decay of the derivatives of a function  $u$  in  $C^{2,\lambda}$  which itself decays exponentially.

**Lemma 3.2** *Let  $\Omega$  be either the whole space  $\mathbb{R}^n$  or the complement of a closed ball with centre 0. Let  $h : \Omega \rightarrow \mathbb{R}$  be a  $C^2$  function, such that  $h$  and its second derivative are bounded over  $\Omega$ . Then the first derivative of  $h$  is bounded, and there exists a constant  $C$ , only depending on the space dimension  $n$ , such that*

$$\|Dh\|_\infty \leq C \|h\|_\infty^{1/2} \|D^2h\|_\infty^{1/2}$$

where  $\|\cdot\|_\infty$  denotes the uniform norm on  $\Omega$ .

**Proof.** Let  $x = (x_1, \dots, x_n) \in \Omega$ ,  $1 \leq i \leq n$  and  $r > 0$ . If  $\Omega$  is the complement of a closed ball, one of the half-lines starting at the point  $x$  and going to infinity following the direction  $x_i$  lies completely in  $\Omega$ . Therefore, either  $x + 2re_i$  or  $x - 2re_i$  is in  $\Omega$ , where  $e_i$  is the  $i$ -th vector of the canonical basis in  $\mathbb{R}^n$ . If  $\Omega = \mathbb{R}^n$ , the whole line lies of course in  $\Omega$ . We can therefore use Taylor's formula to get

$$h(x_1, \dots, x_i \pm 2r, \dots, x_n) = h(x) \pm 2r \frac{\partial h}{\partial x_i}(x) + 2r^2 \frac{\partial^2 h}{\partial x_i^2}(\xi)$$

where  $\xi$  is on the segment between  $x$  and  $x \pm 2re_i$ . Thus we get the following inequality :

$$\left| \frac{\partial h}{\partial x_i}(x) \right| \leq \frac{1}{r} \|h\|_\infty + r \|D^2 h\|_\infty$$

which is true for **any**  $r > 0$ . If  $D^2 h$  is identically zero, then the boundedness assumption on  $h$  implies that  $h$  is a constant, and the wanted result is trivially true. Otherwise, we choose

$$r = \sqrt{\frac{\|h\|_\infty}{\|D^2 h\|_\infty}}$$

and so

$$\left| \frac{\partial h}{\partial x_i}(x) \right|^2 \leq 4 \|h\|_\infty \|D^2 h\|_\infty$$

If we sum from 1 to  $n$ , we get :

$$\|Dh(x)\|^2 \leq 4n \|h\|_\infty \|D^2 h\|_\infty$$

and therefore our interpolation lemma.

The following is the main result of this section.

**Theorem 3.3** *There exists  $\varepsilon_1 > 0$  such that for  $0 < \varepsilon < \varepsilon_1$  the solution  $u_\varepsilon$  given by theorem 3.1 has the following properties:*

1. *It is in the Hölder space  $C^{2,\lambda}(\mathbb{R}^n)$ .*
2. *It and its first three derivatives have exponential decay at infinity.*
3. *It is a non-degenerate solution of  $-\Delta u + F(V(\varepsilon x), u) = 0$  in the sense of Hölder spaces.*

**Proof.** (1) If we can show that  $u_\varepsilon \in L^\infty$  it will follow that  $\Delta u_\varepsilon = F(V(\varepsilon x), u) \in L^\infty$  and so  $u_\varepsilon \in C^1(\mathbb{R}^n)$  as observed in the proof of theorem 2.1. It will then follow that  $g(x) := F(V(\varepsilon x), u) \in C^1(\mathbb{R}^n)$  (using the boundedness of  $D_a F$ ,  $D_u F$  and  $DV$ ) and hence  $\Delta u \in C^1(\mathbb{R}^n)$ . We then deduce  $u \in C^{2,\lambda}$ .

To show that  $u_\varepsilon \in L^\infty$  we first take the case  $n < 4$ . Then  $H^2 \subset L^\infty$  so that  $u_\varepsilon \in L^\infty$ .

Next assume  $n = 4$ . Then  $u_\varepsilon \in L^r$  for  $2 \leq r < \infty$  by the Sobolev embeddings and also  $\Delta u_\varepsilon = g \in L^r$  for  $2 \leq r < \infty$ . Hence  $u_\varepsilon \in W^{2,r}$  for  $2 \leq r < \infty$  so  $u_\varepsilon \in L^\infty$  since  $W^{2,r} \subset L^\infty$  for  $r > 2$ .

For  $n = 5$  recall that  $1 \leq \alpha_1 < 5$ . Suppose that  $u_\varepsilon \in W^{2,r}$  for some  $r$  in the range  $2 \leq r < \frac{5}{2}$ . Then the Sobolev embedding gives  $u_\varepsilon \in L^2 \cap L^{5r/(5-2r)}$ . Moreover

$$|\Delta u_\varepsilon| \leq C(|u_\varepsilon| + |u_\varepsilon|^{\alpha_1}) \in L^{\frac{5r}{(5-2r)\alpha_1}}$$

Since also  $u_\varepsilon \in L^{5r/(5-2r)\alpha_1}$  we find that  $u_\varepsilon \in W^{2,5r/(5-2r)\alpha_1}$ . But the iteration (“bootstrap”)

$$r_{k+1} = \frac{5r_k}{(5-2r_k)\alpha_1}, \quad r_0 = 2$$

will lead, after a finite number of steps, to  $r_k > 5/2$  (this would not work if  $\alpha_1 = 5$ ). So  $u_\varepsilon \in W^{2,r}$  for some  $r > 5/2$  and this gives  $u_\varepsilon \in L^\infty$ .

Note that for this part of the proof we needed neither the polynomial growth of the second derivatives of  $V$  nor the positivity condition on  $F$ . Both are needed in the sequel.

(2) By the conclusion of (1)  $u_\varepsilon$  is uniformly continuous. Since also  $u_\varepsilon \in L^2$  we see that  $\lim_{|x| \rightarrow \infty} u_\varepsilon(x) = 0$ . We can now deduce that  $u_\varepsilon$  has exponential decay most easily by applying [6, theorem 2.2]. Its premises hold because  $F$  is continuously differentiable,  $F(x, 0) = 0$ ,  $u_\varepsilon$  is continuous and decaying, and

$$0 < h < D_u F(x, 0) < H$$

for all  $x$ .

By differentiating  $g$  twice, using growth conditions on  $F$ , exponential decay of  $u_\varepsilon$  as proved above, and polynomial growth of  $D^2V$ , we see that  $D^2g$  is globally bounded; hence  $g \in C^{1,\lambda}$ . Another application of the elliptic regularity principle gives  $u_\varepsilon \in C^{3,\lambda}$ .

Now  $u_\varepsilon$  has exponential decay, and we have bounds on the first three derivatives. So

$$|u_\varepsilon(y)| \leq D e^{-\mu|y|}$$

for absolute constants  $D$  and  $\mu$ , and every  $y \in \mathbb{R}^n$ . Now let  $x \in \mathbb{R}^n$  and  $R < |x|$ ; using the interpolation lemma 3.2 in  $\Omega = \mathbb{R}^n \setminus \overline{B(0, R)}$ , we find

$$\begin{aligned} \|Du_\varepsilon(x)\| &\leq C \|u_\varepsilon\|_{\infty, \Omega} \|D^2u_\varepsilon\|_{\infty, \Omega} \\ &\leq C D e^{-\mu R} \|D^2u_\varepsilon\|_{\infty, \mathbb{R}^n} \\ &\leq C' e^{-\mu R} \end{aligned}$$

for constants  $C'$  and  $\mu$  independent of  $x$  and  $R$ ; this holds for any  $R > |x|$  and so

$$\|Du_\varepsilon(x)\| \leq C' e^{-\mu|x|}$$

hence  $Du_\varepsilon$  decays exponentially. Now, we can use this decay and the bound on the third derivative of  $u_\varepsilon$  to get exponential decay of  $D^2u_\varepsilon$  as well by exactly the same argument.

Now we differentiate  $g$  a third time. Using the exponential decay of  $u_\varepsilon$  and  $Du_\varepsilon$  as just established, polynomial growth of  $D^2V$  and  $D^3V$ , and the growth conditions on  $F$  and its derivatives, we find that  $D^3g$  is also globally bounded, and so, by the same argument,  $u_\varepsilon \in C^{4,\lambda}$  and the third derivative of  $u_\varepsilon$  decays exponentially.

(3). It follows from theorem 3.1 that the solution  $u_\varepsilon$  is non-degenerate in the sense of Sobolev spaces. It is also in  $C^{2,\lambda} \cap H^2$ , and decays at infinity. Since  $F$  satisfies the growth conditions and the positivity assumption needed by theorem 2.1, it is therefore also non-degenerate in the sense of Hölder spaces.

## 4 Application to multibumps

Our object in this section is to use non-degeneracy in the sense of Hölder spaces of the solutions provided by theorem 3.1, and their exponential decay, as proved in the last section, to produce multibump solutions, in particular, solutions with infinitely many bumps. In order to do this we shall apply a technique devised by Angenent [2] the ingredients of which are as follows:

(A1) An equation

$$-\Delta u + G(x, u) = 0$$

in which  $G$  is three times continuously differentiable, and is bounded uniformly in  $x$  for  $u$  in bounded sets.

(A2) A uniformly locally finite covering of  $\mathbb{R}^n$  by open sets  $V_a$ ,  $a \in A$ .

(A3) A partition of unity  $f_a$ ,  $a \in A$ , subordinate to  $V_a$ .

(A4) A family of solutions  $u_a$  in  $C^{2,\lambda}(\mathbb{R}^n)$  indexed by  $A$  and non-degenerate in the sense of Hölder spaces.

(A5) The operators  $A_a = -\Delta + D_u F(x, u_a(x)) : C^{2,\lambda} \rightarrow C^{0,\lambda}$ .

(A6) A collection of quantities,  $M_u$ ,  $M_V$ ,  $K$  and  $R$ , assumed to be finite and satisfying:

$$\begin{aligned} M_u &= \sup_{a \in A} \|u_a|_{V_a}\|_{C^{2,\lambda}(\overline{V_a})} \\ M_V &= \max_{a \in A} |\{b \in A : V_a \cap V_b \neq \emptyset\}| \\ K &= \sup_{a \in A} \|A_a^{-1}\| \\ R &= \left( \sup_{\substack{a \in A \\ x \in \mathbb{R}^n \\ |\alpha|=3}} \|D^\alpha f_a(x)\| \right)^{-1/3} \end{aligned}$$

The following result can be found in [2]:

**Theorem 4.1** *Given the assumptions contained in (A1)–(A6) there exists  $R^*$ , depending only on  $M_u$ ,  $M_V$  and  $K$ , such that if  $R \geq R^*$  the following holds: For every  $\eta > 0$  there exists  $\delta > 0$  such that if*

$$\|(u_a - u_b)|_{V_a \cap V_b}\|_{C^{2,\lambda}(\overline{V_a \cap V_b})} \leq \delta$$

for all  $a, b \in A$  such that  $V_a \cap V_b \neq \emptyset$ , then there exists a solution  $u \in C^{2,\lambda}(\mathbb{R}^n)$  of  $-\Delta u + G(x, u) = 0$ , such that

$$\|(u - u_a)|_{V_a}\|_{C^{2,\lambda}(\overline{V_a})} \leq \eta$$

for all  $a \in A$ .

Broadly speaking, the theorem asserts that the restrictions of given solutions  $u_a$  to the sets  $V_a$  can be glued together approximately to create a new solution  $u$  provided the overlaps of  $u_a$  on  $V_a$  and  $u_b$  on  $V_b$  are suitably small. Angenent himself views this as a version of the shadowing lemma of dynamical systems theory, an attractive analogy which is not made very precise in his paper.

This theorem is not as general as the main result of [2], in which the solutions  $u_a$  are only given locally on  $V_a$ , and defined to be non-degenerate in a rather complex and hard to verify sense. In an example given in the paper the result as we have stated it is given as an important special case. We also note that the role of  $R$  is a bit peculiar: the partition of unity could be varied without changing the covering, thus changing  $R$  but obviously not affecting whether  $u$  exists or not.

We wish to apply theorem 4.1 in the previous context of the non-linear Schrödinger equation, using the non-degeneracy information given by theorem 3.3 to produce infinitely many bumps. To do so, infinitely many critical points of the potential  $V$  are needed; we assume from now on that  $V$  is periodic. Although this assumption is restrictive, it facilitates a fairly straightforward, but non-trivial, application of theorems 3.3 and 4.1.

**Theorem 4.2** *Let  $F$  and  $V$  satisfy S1–S4 [see section 3] and let  $\varepsilon_1$  be as in the conclusion of theorem 3.3. Assume that  $V$  is periodic with period lattice  $\mathbb{Z}^n$ . Let  $\varepsilon < \varepsilon_1$  and  $\eta > 0$ . Then there exists  $r_0$  such that, if  $r$  is an integer such that  $r > r_0$ , and if  $A$  is any subset of  $\varepsilon^{-1}r\mathbb{Z}^n$ , then there exists a solution  $U \in C^{2,\lambda}$  of  $-\Delta u + F(V(\varepsilon x), u) = 0$  such that*

$$\left\| U - \sum_{a \in A} u_\varepsilon(\cdot - a) \right\|_{C^{2,\lambda}(\mathbb{R}^n)} \leq \eta \quad (4.1)$$

**Proof.** The first part of the proof leading to the construction of the solution  $U$  is based on an example sketched in [2]. Throughout the proof  $\varepsilon$  is a fixed number satisfying  $\varepsilon < \varepsilon_1$ .

We begin by defining the ingredients for theorem 4.1. First, we set up a uniformly locally finite covering of  $\mathbb{R}^n$  and a partition of unity. Let  $r$  be a fixed integer; we pick a subset  $A_0$  of the lattice  $\mathbb{Z}^n$  and let  $A = r\varepsilon^{-1}A_0$ . Then  $A$  is a subset of the lattice  $\Gamma = r\varepsilon^{-1}\mathbb{Z}^n$ , itself a sublattice of the lattice  $\varepsilon^{-1}\mathbb{Z}^n$  of periods of  $V(\varepsilon x)$  under which the problem is invariant. The number  $r$  will be increased in the course of the proof, but independently of the subset  $A_0$ .

For each  $a \in \Gamma$ , let  $V_a$  be the hypercube in  $\mathbb{R}^n$  centred at  $a$ , of side  $3r/2\varepsilon$ . Obviously the  $V_a$  form a uniformly locally-finite covering of  $\mathbb{R}^n$  and the quantity  $M_V$  introduced in (A6) equals  $3^n$ . We will denote by  $W_a \subset V_a$  the hypercube centered at  $a$ , of side  $r/2\varepsilon$ .

A partition of unity is naturally defined by choosing a  $C^\infty$  function  $\varphi$  supported in the “unit” hypercube centered at 0 of side  $3/2$  such that

$$\sum_{k \in \mathbb{Z}^n} \varphi(x + k) = 1$$

We then rescale it by the formula

$$f_a(x) = \varphi\left(\frac{\varepsilon}{r}(x - a)\right), \quad a \in \Gamma$$

Each  $f_a$  is supported in  $V_a$ , and they satisfy  $\sum_{a \in \Gamma} f_a(x) = 1$  pointwise. By the definition of  $f_a$  we see that

$$\|D^\beta f_a\|_\infty = \left(\frac{\varepsilon^3}{r^3}\right) \|D^\beta \varphi\|_\infty$$

when  $|\beta| = 3$ . Therefore the quantity  $R$  of (A6) satisfies

$$R = Cr$$

where  $C$  is an absolute constant, depending only on  $n$  and  $\varepsilon$  (which are both fixed in the proof).

We now define the solutions  $u_a$ ,  $a \in \Gamma$ , by

- $u_a = u_\varepsilon(\cdot - a)$  if  $a \in A$ .
- $u_a = 0$  if  $a \in \Gamma \setminus A$

Theorem 3.3 together with the invariance of the problem under translations by elements of  $\Gamma$  ensure the non-degeneracy in the sense of Hölder spaces of the solutions  $u_a$  for  $a \in A$ . Translation invariance also implies that the operators  $A_a^{-1}$ , where  $A_a$  is introduced in (A5), all have the same norm for  $a \in A$ . The non-degeneracy of  $u = 0$  follows from the same argument used to prove the invertibility of the operator  $T$  in the proof of theorem 2.1. The quantity  $M_u$  increases with  $r$  but eventually becomes constant owing to the decay of  $u_\varepsilon$  and its derivatives up to order 3. We assume  $r$  to be chosen sufficiently large to ensure this. Hence the quantities  $M_u$ ,  $M_V$  and  $K$  introduced in (A6) are all independent of  $r$ .

Using the fact that  $R = Cr$  and applying theorem 4.1 we deduce that there exists  $r_1 > 0$  such that for all integers  $r > r_1$  the following is guaranteed: for each  $\rho > 0$  there exists  $\delta > 0$  such that if

$$\|(u_a - u_b)|_{V_a \cap V_b}\|_{C^{2,\lambda}(\overline{V_a \cap V_b})} \leq \delta$$

for all  $a, b \in \Gamma$  such that  $V_a \cap V_b \neq \emptyset$ , then there exists a solution  $U \in C^{2,\lambda}(\mathbb{R}^n)$ , such that

$$\|(U - u_a)|_{V_a}\|_{C^{2,\lambda}(\overline{V_a})} \leq \varrho$$

for all  $a \in \Gamma$ . But by the decay of  $u_\varepsilon$  and its derivatives up to order 3 it is clear that for all sufficiently large  $r$ , say  $r \geq r_2$ , we have

$$\|u_\varepsilon\|_{C^{2,\lambda}(\mathbb{R}^n \setminus W_0)} \leq \delta/2$$

where  $W_0$  is the hypercube centred at 0, of side  $r/2\varepsilon$ . For such  $r$  it follows that

$$\|u_\varepsilon(\cdot - a) - u_\varepsilon(\cdot - b)\|_{C^{2,\lambda}(\overline{V_a \cap V_b})} \leq \delta$$

and

$$\|u_\varepsilon(\cdot - a)\|_{C^{2,\lambda}(\overline{V_a \cap V_b})} \leq \delta$$

provided  $a \neq b$ . Therefore for  $r \geq r_3 := \max(r_1, r_2)$  there exists a solution  $U \in C^{2,\lambda}$  satisfying

- $\|U\|_{C^{2,\lambda}(\overline{V_a})} \leq \varrho$  for  $a \in \Gamma \setminus A$ .
- $\|U - u_\varepsilon(\cdot - a)\|_{C^{2,\lambda}(\overline{V_a})} \leq \varrho$  for  $a \in A$ .

This completes the construction of the solution  $U$ . It will be noted that exponential decay of  $u_\varepsilon$  was not used, merely decay. We proceed to establish the estimate (4.1) for which the exponential nature of the decay is crucial.

First of all, the series

$$h(x) := \sum_{a \in A} u_\varepsilon(x - a)$$

converges uniformly on every compact subset of  $\mathbb{R}^n$ , because of the exponential decay of  $u_\varepsilon$ . A similar result holds for

$$\sum_{a \in A} D^\beta u_\varepsilon(x - a)$$

for  $|\beta| \leq 3$ , and since this series represents  $D^\beta h$  in the sense of distributions we conclude that  $h$  belongs to the space  $C^3(\mathbb{R}^n)$ .

Let us estimate  $U - h$  in the  $C^2$  norm. Let  $x \in \mathbb{R}^n$ , and pick  $a_0 \in \Gamma$  so that  $x \in V_{a_0}$ . Then

$$|U(x) - h(x)| \leq |U(x) - \xi_{a_0} u_\varepsilon(x - a_0)| + \sum_{a \in A, a \neq a_0} |u_\varepsilon(x - a)|$$

where  $\xi_{a_0} = 1$  if  $a_0 \in A$ , and  $\xi_{a_0} = 0$  if not. In both cases, the first term is smaller than  $\varrho$ . We shall show that the second term is also small owing to the exponential decay of  $u_\varepsilon$ , provided  $r$  is large enough. First of all we bound it by the same sum over the whole lattice  $\Gamma$  minus the point  $a_0$ . Then by translating the whole lattice we can assume  $a_0 = 0$ , and  $x$  lies in  $V_0$ . Because of the exponential decay of  $u_\varepsilon$ , this sum is smaller than

$$C \sum_{a \in \Gamma, a \neq 0} e^{-\mu|x-a|_1} = C \sum_{b \in \mathbb{Z}^n, b \neq 0} e^{-\mu|x-\frac{rb}{\varepsilon}|_1}$$

for positive constants  $C$  and  $\mu$ , where  $|\cdot|_1$  is the  $l^1$  norm in  $\mathbb{R}^n$ , that is,  $|(x_1, \dots, x_n)|_1 = \sum_i |x_i|$ . Therefore we can separate the coordinates, and this is equal to

$$C \prod_{i=1}^n \sum_{k_i \in \mathbb{Z}} e^{-\mu|x_i - \frac{rk_i}{\varepsilon}|} - C e^{-\mu|x|_1}$$

where  $x = (x_1, \dots, x_n)$ . Remember that for each  $i$   $|x_i| \leq 3r/4\varepsilon$  by definition of  $V_0$ . We fix  $1 \leq i \leq n$ , and write  $k$  for  $k_i$ ; we have

$$\begin{aligned} \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} e^{-\mu|x_i - \frac{rk}{\varepsilon}|} &= \sum_{k=1}^{+\infty} e^{\mu(x_i - \frac{r}{\varepsilon}k)} + \sum_{k=-\infty}^{-1} e^{\mu(\frac{r}{\varepsilon}k - x_i)} \\ &= (e^{\mu x_i} + e^{-\mu x_i}) \frac{e^{-\frac{\mu r}{\varepsilon}}}{1 - e^{-\frac{\mu r}{\varepsilon}}} \\ &\leq 2e^{\frac{3\mu r}{4\varepsilon}} \frac{e^{-\frac{\mu r}{\varepsilon}}}{1 - e^{-\frac{\mu r}{\varepsilon}}} \\ &= \frac{2e^{-\frac{\mu r}{4\varepsilon}}}{1 - e^{-\frac{\mu r}{\varepsilon}}} \end{aligned}$$

and from this we deduce

$$\begin{aligned} \sum_{a \in A, a \neq a_0} |u_\varepsilon(x - a)| &\leq C \prod_{i=1}^n \left( \frac{2e^{-\frac{\mu r}{4\varepsilon}}}{1 - e^{-\frac{\mu r}{\varepsilon}}} + e^{-\mu|x_i|} \right) - Ce^{-\mu|x|} \\ &\leq C' \frac{e^{-\frac{\mu r}{4\varepsilon}}}{1 - e^{-\frac{\mu r}{\varepsilon}}} \end{aligned}$$

provided  $r$  is chosen large enough so that  $e^{-\frac{\mu r}{4\varepsilon}}/(1 - e^{-\frac{\mu r}{\varepsilon}}) < 1$ , and where the constant  $C'$  depends only on  $C$  and  $n$ .

Now the right-hand side tends to 0 (exponentially fast) as  $r$  tends to infinity, so we can choose  $r_4 \geq r_3$  (independently of  $A_0$  and  $a_0$ ) so that for  $r \geq r_4$

$$\sum_{a \in A, a \neq a_0} |u_\varepsilon(x - a)| \leq \varrho$$

if  $x \in V_a$ , and therefore,  $|h(x) - U(x)| \leq 2\varrho$ , for each  $x \in \mathbb{R}^n$ . Very similar considerations hold for the first two derivatives (using lemma 3.3) so that we may choose  $r_4$  so that

$$\|h - U\|_{C^2(\mathbb{R}^n)} \leq 6\varrho$$

for  $r \geq r_4$ .

Finally we estimate the Hölder semi-norm. Take  $x, y \in \mathbb{R}^n$ . The objective is the inequality

$$(*) := |D^2U(x) - D^2U(y) - D^2h(x) + D^2h(y)| \leq 2\varrho|x - y|^\lambda$$

Firstly, if  $|x - y| \geq 1$ , we simply write

$$(*) \leq 2\|D^2U - D^2h\|_\infty \leq 2\varrho \leq 2\varrho|x - y|^\lambda,$$

so the only difficulty comes from the case where  $|x - y| \leq 1$ . In that case,  $x$  and  $y$  belong to the same component of the covering, say  $V_{a_0}$ . We split  $(*)$  as we did before

$$\begin{aligned} (*) &\leq |D^2U(x) - D^2U(y) - \xi_{a_0}(D^2u_\varepsilon(x - a_0) - D^2u_\varepsilon(y - a_0))| \\ &\quad + \sum_{a \neq a_0} |D^2u_\varepsilon(x - a) - D^2u_\varepsilon(y - a)| \end{aligned}$$

Because of the properties of  $U$  the first term is smaller than  $\varrho|x - y|^\lambda$ . To take care of the second term, we use the mean value theorem. For each  $a \neq a_0$  it yields the existence of  $\theta_a$ , with  $|\theta_a| \leq 1$ , such that  $|D^2u_\varepsilon(x - a) - D^2u_\varepsilon(y - a)| \leq |D^3u_\varepsilon(x + \theta_a - a)||x - y|$ . The bound on  $|\theta_a|$  arises from  $|(x - a) - (y - a)| \leq 1$ . It then follows that

$$(*) \leq \varrho|x - y|^\lambda + C|x - y| \sum_{a \neq a_0} e^{-\mu|x + \theta_a - a|}$$

But since  $|x - y| \leq |x - y|^\lambda$ , and  $|x + \theta_a - a| \geq |x - a| - 1$

$$(*) \leq \left( \varrho + Ce^\mu \sum_{a \neq a_0} e^{-\mu|x-a|} \right) |x - y|^\lambda$$

Now we can use our previous calculations to find  $r_5 > r_4$  such that

$$(*) \leq 2\rho|x - y|^\lambda$$

when  $|x - y| \leq 1$  and  $r \geq r_5$ , so that :

$$\|h - U\|_{C^{2,\lambda}} \leq 8\rho.$$

Finally, for  $\eta$  as defined in the theorem we set  $\varrho = \eta/8$ ; then the corresponding solution  $U$  is the required multi-bump solution.

**Remark.** It follows from the proof that this theorem produces uncountably many multi-bump solutions. Indeed, pick  $\eta$  such that

$$\eta \leq \frac{2}{3}\|u_\varepsilon\|_{L^\infty}$$

and for each  $r \geq r_0$  and  $A \subset \varepsilon^{-1}r\mathbb{Z}^n$  let  $U_A$  be a solution as provided by theorem 4.2. Then from  $U := U_A$  we may retrieve  $A$  as

$$A = \left\{ a \in \varepsilon^{-1}r\mathbb{Z}^n, \max_{V_a} |U| \geq \frac{2}{3}\|u_\varepsilon\|_{L^\infty} \right\}$$

because if  $U = U_B$ , let  $a \in A$ ,  $a \notin B$  (the other case is treated similarly), so that  $|U(x)| \geq 2/3\|u_\varepsilon\|_\infty$  for one  $x$  in  $V_a$ , but also

$$\|U\|_{C^{2,\lambda}(V_a)} \leq \frac{\eta}{2} \leq \frac{1}{3}\|u_\varepsilon\|_\infty,$$

which is a contradiction. Similarly  $B \subset A$ , and  $\psi$  is one-to-one.

One can also see that this theorem produces uncountably many geometrically different solutions; recall that two solutions  $u$  and  $v$  are geometrically different if  $u$  is not contained in the orbit of  $v$  under the natural action of  $\varepsilon^{-1}\mathbb{Z}^n$  by translation. This follows directly from the fact that this action by translation on the power set  $\mathcal{P}(\varepsilon^{-1}r\mathbb{Z}^n)$  has uncountably many orbits; then if we pick one multibump solution for each  $a$  in a system of representatives of those orbits, they will be geometrically different.

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