A scaling approach to bumps and multi-bumps for non-linear partial differential equations

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Abstract

The problem $-\triangle u+F(V(\epsilon x),u)=0$ is considered in $\mathbf{R}^n$. For small $\epsilon>0$ solutions are obtained that approach, as $\epsilon\to 0$, a linear combination of specified functions, mutually translated by $O(1/\epsilon)$. These are the so-called multi-bump solutions. The method involves a rescaling of the variables and the use of a modified implicit function theorem. The usual implicit function theorem is inapplicable owing to lack of convergence of the derivative of the non-linear Hilbert space operator, obtained after an appropriate rescaling, in the operator-norm topology. An asymptotic formula for the solution for small ϵ is obtained.

1 Statement of the problem

Consider the problem

$$-\triangle u+F(a,u)=0 \tag{1.1}$$

posed in a space of real functions E on $\mathbf{R}^n$, where F is a differentiable function of u and a is a real parameter in some interval I. Suppose that for each a we know a solution of (1), $\phi_a(x)$, possessing some non-degeneracy. This requires, for example, that the linearized operator $-\triangle+\frac{\partial F}{\partial u}(a,\phi_a(x))$ is a Fredholm operator of index 0 (with suitable choice of codomain) and that its kernel is spanned by the n partial derivatives $D_j\phi_a(x)$.

Problem 1.1 *Find solutions in E when a is replaced by a* function of x *that is "nearly constant". In particular how are the solutions related to the functions $\phi_a(x)$ and their translates?*

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In this paper we shall concentrate entirely on the case

$$-\triangle u + F(V(\epsilon x), u) = 0 \tag{1.2}$$

where $\epsilon > 0$ is a small parameter and V a given function with range in I.

Example 1.2 *The non-linear Schrödinger equation. Take* $F(a,u) = au - u^p$, $I =]0,\infty[$, E *is the Hilbert space of real functions* $W^{2,2}(\mathbf{R}^n)$. *This gives the problem of semiclassical states of the non-linear Schrödinger equation with potential* $V(x)$, *equivalent to*

$$-\epsilon^2 \triangle_y u + V(y)u - u^p = 0 \tag{1.3}$$

where $y = \epsilon x$. *The function* $\phi_a(x)$ *is the spherically symmetric, positive solution (the ground state) of* $-\triangle u + au - u^p = 0$, *which exists for all* $p > 1$ *if* $n = 1, 2$, *and for* $1 < p < \frac{n+2}{n-2}$ *if* $n > 2$.

There are of course other ways to replace a by a nearly constant function, and though they fall outside the scope of this article we mention one here.

Example 1.3 *The non-linear eigenvalue problem. Take* $F(a,u) = u - au^p$. *Replace* a *by the function* $a + h(\lambda^{-\frac{1}{2}}x)$, *where* $\lambda > 0$ *small and* h *is in some sense small at infinity. The unperturbed problem has the solution* $a^{\frac{1}{1-p}}\phi_1(x)$ *where* ϕ_a *was defined above. The problem is equivalent to*

$$\triangle_y v + (a + h(y))v^p = \lambda v$$

where $x = \sqrt{\lambda} y$, $v = \lambda^{\frac{1}{p-1}} u$.

Equation (1.2) has been extensively treated in the special case of the non-linear Schrödinger equation (1.3). In 1986 Floer and Weinstein [5] showed the existence of solutions of the 1-dimensional case of (1.3) that accumulate near a non-degenerate critical point of V for small ϵ. In 1990 Y. G. Oh [13] showed the existence of solutions that accumulate near several critical points (multibump solutions). These papers use an initial reduction to finite dimensions by a variant of the Liapunov-Schmidt method followed by solving the finite-dimensional equations by the contraction mapping principle. The reduction has to be carried out at an approximate solution. In essence this is a way to circumvent the same difficulty as Theorem 2.1 of the present paper is designed to do. In [11] a simpler approach was proposed. In these papers the $W^{2,2}$ topology is used.

More recently problem (1.3) has been widely treated as variational, which requires the use of the coarser $W^{1,2}$ topology. Against this there is no need for non-degeneracy of the critical points of V (see our Theorems 3.8 and 4.3), although some of the results seem to use only minima. The methods used owe much to those introduced in [3] for a problem involving periodic differential equations. See for example [14], [15], [16], [6], [2], [9], [10].

The treatment of (1.2) proposed in this paper has several features:

1. The $W^{2,2}$ topology is used.
2. Scaling is used with a view to finding an asymptotic development of the solution for small ϵ, with, as far as possible, a non-vanishing principal part.
3. The two steps used in previous treatments, reduction to finite dimensions and the solving of the finite-dimensional equations, are combined into a single step.
4. An unconventional form of the implicit function theorem must be used as the hypotheses of the conventional version of this theorem do not hold.
5. On the negative side, differentiability has to be built in to make Theorem 2.1 work. This means more restrictions on the function $V(x)$ than in some earlier treatments.

Scaling (more precisely rescaling) means in this context an ϵ-dependent change of the state variable u that becomes singular when $\epsilon = 0$. The technique was in effect introduced into bifurcation theory by the seminal paper on bifurcation at a simple eigenvalue [4]. The failure of the implicit function theorem due to discontinuity of the derivative of the non-linear operator in the operator-norm topology was pointed out in [11] and a remedy proposed. In this paper we use the version of the remedy set out in [12] and refer to it as the modified implicit function theorem.

2 The modified implicit function theorem

The theorem set out in [12] (after a preliminary version in [11]) is the following, which we shall sometimes refer to by the acronym "MIFT".

Theorem 2.1 *Let E and F be real Banach spaces, and let $f : \mathbf{R}_+ \times E \to F$. Assume that*

(1) $f(\epsilon, \cdot)$ *is* C^1 *for each* $\epsilon \geq 0$*;*
(2) *there exists* $x_0 \in E$ *such that* $f(0, x_0) = 0$*;*
(3) $D_x f(0, x_0)$ *is invertible;*
(4) $\lim_{\epsilon \to 0^+} f(\epsilon, x_0) = 0$*;*
(5) *for all sufficiently small* $\epsilon > 0$ *the operator* $D_x f(\epsilon, x_0)$ *is invertible and* $||D_x f(\epsilon, x_0)^{-1}||$ *is uniformly bounded as* $\epsilon \to 0^+$*;*
(6) $\lim_{\epsilon \to 0^+, x \to x_0} ||D_x f(\epsilon, x) - D_x f(\epsilon, x_0)|| = 0.$

Then there exist $\epsilon_0 > 0$ *and a neighbourhood* U *of* x_0 *in* E *such that for each* ϵ *in the range* $0 < \epsilon < \epsilon_0$ *there exists a unique solution* $x = x_\epsilon$ *of* $f(\epsilon, x) = 0$ *in* U*. Moreover* $\lim_{\epsilon \to 0} x_\epsilon = x_0$*.*

If, furthermore, $f(\epsilon, x)$ *is a continuous function of* ϵ *for* $\epsilon > 0$ *and for all* x *in a neighbourhood of* x_0*, and the map* $\epsilon \mapsto D_x f(\epsilon, x_0)$ *is continuous in the* strong operator topology*, then the solution* x_ϵ *depends continuously on* ϵ*.*

The real difficulty is in verifying condition (5). A Fredholm operator of index 0 has the property that it is surjective if and only if it is injective; so the verification of (5) is facilitated if $D_x f(\epsilon, x_0)$ is such an operator. In practice we shall verify (5) by applying the following proposition, the proof of which is straightforward.

Lemma 2.2 *Let $\{T_\epsilon\}_{0<\epsilon<\epsilon_0}$, be a family of Fredholm operators of index 0 from E to F. Suppose that there do not exist sequences $\epsilon_\nu \to 0$, $x_\nu \in E$ such that $||x_\nu|| = 1$ and $T_{\epsilon_\nu} x_\nu \to 0$. Then there exists $\epsilon_1 > 0$ such that for all ϵ in the range $0 < \epsilon < \epsilon_1$ the operator T_ϵ is invertible and there exists a constant K independent of ϵ such that $||T_\epsilon^{-1}|| < K$.*

3 Single bump solutions

We set out the conditions that F, V and ϕ_a are supposed to satisfy, interspersed with some useful lemmas. Each lemma assumes the conditions previously introduced. In this section we avoid imposing growth conditions on F, or other conditions that are pointwise w.r.t. x (except (F1) and (F6) below). Instead we list functional analytic properties of the Nemytskii operator induced by F. Note that L^p denotes the space $L^p(\mathbf{R}^n)$, H^k the space $W^{k,2}(\mathbf{R}^n)$, $\mathcal{L}(X,Y)$ the space of bounded linear operators from the Banach space X to the Banach space Y, and $\mathcal{L}_2(X_1 \times X_2, Y)$ the space of bounded bilinear operators from $X_1 \times X_2$ to Y. The H^k-norm of u is denoted by $||u||_{k,2}$ whilst its L^p-norm is denoted by $||u||_p$. Norm symbols without subscripts will usually be operator norms, except when applied to vectors in $\mathbf{R}^n$ where they denote the Euclidean norm.

The conditions imposed may appear rather sweeping; indeed it is plausible that the existence of second derivatives is not needed to make the method work. Our purpose is not to make do with the weakest possible assumptions but to impose reasonable conditions that might be met with in most examples.

Properties of F

(F1) *F possesses continuous second-order partial derivatives.*

(F2) F, $\dfrac{\partial F}{\partial a}$, $\dfrac{\partial^2 F}{\partial a^2}$, $\dfrac{\partial F}{\partial u}$, $\dfrac{\partial^2 F}{\partial u \partial a}$ *and* $\dfrac{\partial^2 F}{\partial u^2}$, *define mappings*

$$\mathbf{F}, \mathbf{F_a}, \mathbf{F_{aa}} : L^\infty \times H^2 \to L^2,$$

$$\mathbf{F_u}, \mathbf{F_{au}} : L^\infty \times H^2 \to \mathcal{L}(H^2, L^2)$$

and

$$\mathbf{F_{uu}} : L^\infty \times H^2 \to \mathcal{L}_2(H^2 \times H^2, L^2),$$

by means of

$$\begin{aligned}
\mathbf{F}(m,u) &= F(m,u),\\
\mathbf{F_a}(m,u) &= \frac{\partial F}{\partial a}(m,u),\\
\mathbf{F_{aa}}(m,u) &= \frac{\partial^2 F}{\partial a^2}(m,u),\\
\mathbf{F_u}(m,u)v &= \frac{\partial F}{\partial u}(m,u)v,\\
\mathbf{F_{au}}(m,u)v &= \frac{\partial^2 F}{\partial u \partial a}(m,u)v,\\
\mathbf{F_{uu}}(m,u)(v_1,v_2) &= \frac{\partial^2 F}{\partial u^2}(m,u)v_1v_2.
\end{aligned}$$

A note on notation. We shall often use the expression $\frac{\partial F}{\partial a}(m,u)$ to denote the function $\frac{\partial F}{\partial a}(m(\cdot),u(\cdot))$ etc. We shall allow ourselves this abuse of notation where there seems to be little scope for misunderstanding. This also saves space in long formulas.

(F3) *The maps* $\mathbf{F}$, $\mathbf{F_a}$, $\mathbf{F_{aa}}$, $\mathbf{F_u}$, $\mathbf{F_{ua}}$ *and* $\mathbf{F_{uu}}$ *have the following* boundedness property. *Each maps bounded subsets of* $L^\infty \times H^2$ *to bounded subsets of the appropriate function or operator space.*

It is known for Nemytskii operators that the mere fact of transforming L^{p_1} into L^{p_2} guarantees continuity and boundedness (see [7]). It therefore seems plausible that (F3) follows at least in part from (F2).

We shall often need the following integration convergence lemma. The parameter ϵ can be replaced by a discrete parameter. In the applications the function $k(\sigma)$ will usually be a constant.

Lemma 3.1 *Let* S *be the unit interval, unit square etc. and let* $f(\epsilon,\sigma,x)$ *be defined in* $\mathbf{R}_+ \times S \times \mathbf{R}^n$ *and* $g(\sigma,x)$ *in* $S \times \mathbf{R}^n$. *Assume that*
(1) *the map* $\sigma \mapsto f(\epsilon,\sigma,\cdot)$ *is continuous from* S *to* L^2*;*
(2) $\lim_{\epsilon\to 0} f(\epsilon,\sigma,\cdot) = g(\sigma,\cdot)$ *in* L^2 *for each* σ*;*
(3) *there exists* $k(\sigma)$ *such that* $||f(\epsilon,\sigma,\cdot)||_2 < k(\sigma)$ *for all* ϵ, σ *and* $\int_S k(\sigma)\,d\sigma < \infty$.
Then the function $h_\epsilon = \int_S f(\epsilon,\sigma,\cdot)\,d\sigma$ *converges in* L^2 *to the function* $h = \int_S g(\sigma,\cdot)\,d\sigma$ *as* $\epsilon \to 0$.

Proof The integral defining h_ϵ can be viewed as that of an L^2-valued function of $\sigma \in S$. Then the result follows from the dominated convergence theorem for such integrals.

Lemma 3.2 **F** *is continuously differentiable and its derivative at* (m,u) *is the linear map*

$$(h,v) \mapsto \mathbf{F_a}(m,u)h + \mathbf{F_u}(m,u)v.$$

Moreover the maps $\mathbf{F_a}$ *and* $\mathbf{F_u}$ *are uniformly continuous on bounded sets.*

Proof We have that

$$\begin{aligned}
&\mathbf{F}(m,u) - \mathbf{F}(m,u_0) - \mathbf{F_u}(m,u_0)(u-u_0) \\
&\qquad = \int_0^1 \left(\frac{\partial F}{\partial u}\Big(m,(1-t)u_0+tu\Big) - \frac{\partial F}{\partial u}\Big(m,u_0\Big)\right)(u-u_0)\,dt \\
&\qquad\qquad = \int_0^1\int_0^1 \frac{\partial^2 F}{\partial u^2}\Big(m,(1-st)u_0+stu\Big)(u-u_0)^2 t\,ds\,dt.
\end{aligned}$$

By condition (F3) the L^2-norm of this function is bounded by a constant times $||u-u_0||_{2,2}^2$. Hence the map $u \mapsto \mathbf{F}(m,u)$ is differentiable with derivative $\mathbf{F_u}(m,u)$.

We also have that

$$\begin{aligned}
&\mathbf{F}(m,u) - \mathbf{F}(m_0,u) - \mathbf{F_a}(m_0,u)(m-m_0) \\
&\qquad = \int_0^1 \left(\frac{\partial F}{\partial a}\Big((1-t)m_0+tm,u\Big) - \frac{\partial F}{\partial a}\Big(m_0,u\Big)\right)(m-m_0)\,dt \\
&\qquad\qquad = \int_0^1\int_0^1 \frac{\partial^2 F}{\partial a^2}\Big((1-st)m_0+stm,u\Big)(m-m_0)^2 s\,ds\,dt.
\end{aligned}$$

By condition (F3) the L^2-norm of this function is bounded by a constant times $||m-m_0||_\infty$. Hence the map $m \mapsto \mathbf{F}(m,u)$ is differentiable with derivative $\mathbf{F_a}(m,u)$.

It now suffices to show that $\mathbf{F_u}$ and $\mathbf{F_a}$ have the uniform continuity property. We have

$$\begin{aligned}
\Big(\mathbf{F_u}(m_2,u_2) - \mathbf{F_u}(m_1,u_1)\Big)w &= \int_0^1 \frac{\partial^2 F}{\partial u^2}\Big(m_2,(1-t)u_1+tu_2\Big)(u_2-u_1)w\,dt \\
&\quad + \int_0^1 \frac{\partial^2 F}{\partial u \partial a}\Big((1-t)m_1+tm_2,u_1\Big)(m_2-m_1)w\,dt.
\end{aligned}$$

If m_1, m_2 are restricted to a bounded set $M \subset L^\infty$ and u_1, u_2 to a bounded set $U \subset H^2$, condition (F3) gives a constant K such that

$$||\mathbf{F_u}(m_2,u_2) - \mathbf{F_u}(m_1,u_1)|| \le K(||m_2-m_1||_\infty + ||u_2-u_1||_{2,2}).$$

A similar argument deals with $\mathbf{F_a}$ and ends the proof of the lemma.

(F4) *Let* $M \subset L^\infty$ *be bounded,* $u, v \in H^2$*, and let* h_ν *be a bounded sequence in* L^∞ *converging pointwise to* 0*. Then*

$$\mathbf{F_a}\big(m,u\big)h_\nu \to 0, \quad \mathbf{F_{aa}}\big(m,u\big)h_\nu \to 0, \quad \mathbf{F_{au}}\big(m,u\big)vh_\nu \to 0$$

in L^2 *uniformly with respect to* $m \in M$*.*

Lemma 3.3 *The operators* $\mathbf{F}$, $\mathbf{F_a}$ *and* $\mathbf{F_u}$ *have the following additional continuity properties. Let* $m_\nu \in L^\infty$ *be a bounded sequence that tends pointwise to* $m \in L^\infty$, *let* $u_\nu \in H^2$ *converge to* $u \in H^2$ *and let* $v \in H^2$. *Then*

$$\mathbf{F}(m_\nu, u_\nu) \to \mathbf{F}(m, u), \quad \mathbf{F_a}(m_\nu, u_\nu) \to \mathbf{F_a}(m, u), \quad \mathbf{F_u}(m_\nu, u_\nu)v \to \mathbf{F_u}(m, u)v.$$

Proof Consider the third limit; the first two are treated similarly. We write

$$\frac{\partial F}{\partial u}(m_\nu, u_\nu)v - \frac{\partial F}{\partial u}(m, u)v = \int_0^1 \frac{\partial^2 F}{\partial u^2}\Big(m_\nu, tu_\nu + (1-t)u\Big)(u_\nu - u)v\, dt \\ + \int_0^1 \frac{\partial^2 F}{\partial u \partial a}\Big(tm_\nu + (1-t)m, u\Big)(m_\nu - m)v\, dt.$$

The two integrands tend to 0 in L^2 for each t as $\nu \to \infty$ because of (F3) and (F4) respectively. Again because of (F3) the L^2-norm of each integrand is bounded by a number independent of t and of ν if the latter is sufficiently large. So by Lemma 3.1 both integrals represent functions that tend to 0 in L^2 as $\nu \to \infty$.

Note that we cannot in general deduce that $\mathbf{F_u}(m_\nu, u_\nu) \to \mathbf{F_u}(m, u)$ in the operator norm under the conditions of Lemma 3.3 unless $m_\nu \to m$ in the L^∞ norm. The conclusion of the lemma is that convergence occurs in the strong operator topology.

(F5) *Let* $w \in H^2$ *and let* v_ν *be a sequence in* H^2 *that converges to* 0 *in the weak topology. Then* $\mathbf{F_{uu}}(m, u)(v_\nu, w) \to 0$ *in* L^2 *uniformly for* (m, u) *in bounded subsets of* $L^\infty \times H^2$.

In the following we view L^2 as embedded in the dual of H^2 by virtue of identifying $v \in L^2$ with the linear functional $H^2 \ni \chi \mapsto \int \chi v\, dx$.

Lemma 3.4 *Let* $m_\nu \in L^\infty$ *be a bounded sequence that tends pointwise to* $m \in L^\infty$ *and let* $u_\nu \in H^2$ *converge weakly to* $u \in H^2$. *Then*

$$\mathbf{F_u}(m_\nu, u_\nu)v_\nu - \mathbf{F_u}(m, u)v_\nu \to 0$$

in the weak topology on the dual of H^2 *for any bounded sequence* $v_\nu \in H^2$.

Proof We must show that

$$\int \Big(\mathbf{F_u}(m_\nu, u_\nu)v_\nu - \mathbf{F_u}(m, u)v_\nu\Big)\chi\, dx \to 0$$

for each $\chi \in H^2$. We write

$$\int \left(\frac{\partial F}{\partial u}\Big(m_\nu, u_\nu\Big) - \frac{\partial F}{\partial u}\Big(m, u\Big)\right) v_\nu \chi\, dx \\ = \int \int_0^1 \frac{\partial^2 F}{\partial u^2}\Big(m_\nu, (1-t)u + tu_\nu\Big)(u_\nu - u)v_\nu \chi\, dt\, dx \\ + \int \int_0^1 \frac{\partial^2 F}{\partial u \partial a}\Big((1-t)m + tm_\nu, u\Big)(m_\nu - m)v_\nu \chi\, dt\, dx.$$

By the Cauchy-Schwarz Lemma this is bounded in absolute value by

$$\|v_\nu\|_2 \cdot \left\| \int_0^1 \frac{\partial^2 F}{\partial u^2}\Big(m_\nu, (1-t)u + tu_\nu\Big)(u_\nu - u)\chi \, dt \right\|_2 + \|v_\nu\|_2 \cdot \left\| \int_0^1 \frac{\partial^2 F}{\partial u \partial a}\Big((1-t)m + tm_\nu, u\Big)(m - m_\nu)\chi \, dt \right\|_2 .$$

Now $||v_\nu||_2$ is bounded and the integrals tend to 0 in L^2 by virtue of (F4) and (F5). (In reading the last displayed expression note that the integrals represent functions of x of which we compute the L^2-norm.)

Lemma 3.5 *Let m_ν be a bounded family in L^∞, u_ν, v_ν, w_ν bounded sequences in H^2, such that either*

(1) *$u_\nu - v_\nu$ is convergent in H^2 and w_ν converges weakly to 0; or*

(2) *$u_\nu - v_\nu$ converges weakly to 0 in H^2 and w_ν is convergent.*

Then

$$\big(\mathbf{F_u}(m_\nu, u_\nu) - \mathbf{F_u}(m_\nu, v_\nu)\big) w_\nu \to 0$$

in L^2. Furthermore $\mathbf{F_u}(m, u) - \mathbf{F_u}(m, v)$ is a compact operator for each $m \in L^\infty$ and $u, v \in H^2$.

Proof

$$\left(\frac{\partial F}{\partial u}(m_\nu, u_\nu) - \frac{\partial F}{\partial u}(m_\nu, v_\nu)\right) w_\nu = \int_0^1 \frac{\partial^2 F}{\partial u^2}\Big(m_\nu, tu_\nu + (1-t)v_\nu\Big)(u_\nu - v_\nu) w_\nu \, dt$$

and the first conclusion follows from (F5). Then $\mathbf{F_u}(m, u) - \mathbf{F_u}(m, v)$ is compact since it maps weakly convergent sequences to norm convergent ones.

A family of functions u_ν is said to have *uniform exponential decay* if there exist $K > 0$ and $k > 0$ such that $|u_\nu(x)| < Ke^{-k||x||}$ for all $x \in \mathbf{R}^n$ and ν.

(F6) *Let m_ν be a bounded sequence in L^∞ and let u_ν have uniform exponential decay. Then the families $\mathbf{F}(m_\nu, u_\nu)$, $\mathbf{F_a}(m_\nu, u_\nu)$ and $\mathbf{F_{aa}}(m_\nu, u_\nu)$ have uniform exponential decay.*

Properties of ϕ_a

The function $\phi_a(x)$ is a solution to $-\triangle u + F(a, u) = 0$ in H^2 and has the following properties. We let $r = ||x||$.

(Φ1) *$\phi_a(x) = \Phi_a(r)$ is spherically symmetric.*

(Φ2) $\displaystyle\int \frac{\partial F}{\partial a}\Big(a, \Phi_a(r)\Big)\Phi'(r) r \, dx \neq 0.$

(Φ3) *ϕ_a and its first derivatives have exponential decay, that is, there exist constants K and k (which may depend on a though not on x) such that $|\phi_a(x)|$, $||\nabla\phi_a(x)|| \leq Ke^{-kr}$.*

(Φ4) *The operator $-\triangle + \frac{\partial F}{\partial u}(a, \phi_a(x)) : H^2 \to L^2$ has as kernel the space spanned by the n partial derivatives $D_j\phi_a(x)$, which are assumed to be independent, and its range is the space orthogonal in L^2 to its kernel.*

Note that the properties (Φ1–4) hold in the model case of Example 1.2 [17], [19], [8].

Properties of V

(V1) *V is C^2 with range in the interval I, it and its first partial derivatives are bounded, whilst its second partial derivatives have polynomial growth.*

Lemma 3.6 *The map $s \mapsto F(V(x+s), u)$ from $\mathbf{R}^n$ to L^2 is differentiable and its derivative is a continuous function of $(s, u) \in \mathbf{R}^n \times H^2$.*

Note. This is not just the chain rule; for one thing the map $s \mapsto V(x+s)$ is not necessarily differentiable from s to L^∞.

Proof Let e_i be the ith basis vector in $\mathbf{R}^n$. We have

$$\begin{aligned}
&F(V(x+s+he_i), u) - F(V(x+s), u) - \frac{\partial F}{\partial a}\Big(V(x+s), u\Big)D_iV(x+s)h \\
&= h\int_0^1 \left[\frac{\partial F}{\partial a}\Big(V(x+s+the_i), u\Big)D_iV(x+s+the_i) - \frac{\partial F}{\partial a}\Big(V(x+s), u\Big)D_iV(x+s)\right] dt \\
&\qquad = h\int_0^1 \frac{\partial F}{\partial a}\Big(V(x+s+the_i), u\Big)\Big(D_iV(x+s+the_i) - D_iV(x+s)\Big)\, dt \\
&\qquad\qquad + h\int_0^1 \left[\frac{\partial F}{\partial a}\Big(V(x+s+the_i), u\Big) - \frac{\partial F}{\partial a}\Big(V(x+s), u\Big)\right] D_iV(x+s)\, dt.
\end{aligned}$$

In the first integral the integrand, regarded as a function of x, tends to 0 in L^2 as $h \to 0$ for each $t \in [0, 1]$ by (F4) and (V1). The L^2-norm of the integrand is uniformly bounded w.r.t t by (F3) and (V1). The integrand of the second integral tends to 0 in L^2 as $h \to 0$ for each $t \in [0, 1]$ by Lemma 3.3. It too has a bound in L^2 independent of t. Hence both integrals tend to 0 in L^2 as $h \to 0$ by Lemma 3.1. This shows that the mapping of the lemma is differentiable and its partial derivative is the element $\frac{\partial F}{\partial a}\big(V(x+s), u\big)D_iV(x+s) \in L^2$. That the map $(s, u) \mapsto \frac{\partial F}{\partial a}\big(V(x+s), u\big)D_iV(x+s)$, from $\mathbf{R}^n \times H^2$ to L^2, is continuous, follows from Lemma 3.3, (V1) and (F4).

Positivity property

We place last a condition satisfied by F and V and needed in order to verify the premises of the modified implicit function theorem.

(P1) There exists $\delta > 0$ such that $\dfrac{\partial F}{\partial u}\big(a, 0\big) > \delta$ for all a in the range of V.

We shall need a version of *Wang's Lemma* (see [18, 12]).

Lemma 3.7 *Let $f_\nu(x)$ be a family of measurable functions such that*

$$0 < \delta < f_\nu(x) < K$$

for all ν and certain constants δ and K. Let μ_ν be a sequence of non-negative numbers and let v_ν be a sequence in H^2 such that

$$-\triangle v_\nu + (f_\nu(x) + \mu_\nu) v_\nu \to 0$$

in L^2. Then $v_\nu \to 0$ in H^2.

The following is the main result of this section.

Theorem 3.8 *Suppose conditions (F1–6), (Φ1–4), (V1) and (P1) are satisfied. Let b be a non-degenerate critical point of V. Then for all sufficiently small $\epsilon > 0$ equation (1.2) has a solution $u_\epsilon(x)$ of the form*

$$u_\epsilon(x) = \phi\Big(x - \frac{b}{\epsilon} + s_\epsilon\Big) + \epsilon^2 w_\epsilon\Big(x - \frac{b}{\epsilon} + s_\epsilon\Big)$$

such that $s_\epsilon \in \mathbf{R}^n$, $w_\epsilon \in H^2$ and is L^2-orthogonal to the functions $D_j\phi_a$ for $j = 1, \ldots, n$, where $a = V(b)$, and s_ϵ and w_ϵ depend continuously on ϵ. Moreover $s_\epsilon \to 0$, and $w_\epsilon \to \eta$ in the H^2-norm, as $\epsilon \to 0^+$, where η is the unique solution orthogonal to the n functions $D_j\phi_a$ of the non-homogeneous linear problem

$$-\triangle v + \frac{\partial F}{\partial u}\Big(a, \phi_a(x)\Big) v = -\frac{1}{2}\frac{\partial F}{\partial a}\Big(a, \phi_a(x)\Big)\big(H(b)x \cdot x\big) \tag{3.1}$$

and $H(x)$ is the Hessian matrix $D_{i,j}V(x)$.

Note. In the last equation we denote by $A \cdot B$ the Euclidean inner product of vectors A and B in $\mathbf{R}^n$.

Proof We solve (1.2) by scaling. Define a *new state variable* (s, w): $s \in \mathbf{R}^n$, $w \in W$ where

$$W = \Big\{w \in H^2 : \int w D_j \phi_a = 0, j = 1, \ldots, n\Big\}.$$

The rescaling is defined by

$$u(x) = \phi_a\Big(x - \frac{b}{\epsilon} + s\Big) + \epsilon^2 w\Big(x - \frac{b}{\epsilon} + s\Big),$$

or for short

$$u(x) = \phi_a(x+\xi) + \epsilon^2 w(x+\xi)$$

where $\xi = -\frac{b}{\epsilon} + s$. Substituting the rescaled variables we obtain

$$-\triangle\phi_a(x+\xi) - \epsilon^2\triangle w(x+\xi) + F\big(V(\epsilon x), \phi_a(x+\xi) + \epsilon^2 w(x+\xi)\big) = 0,$$

and replacing x by $x - \xi$,

$$-\triangle\phi_a - \epsilon^2\triangle w + F\big(V(\epsilon(x-\xi)), \phi_a + \epsilon^2 w\big) = 0.$$

Using $-\triangle\phi_a(x) + F(a, \phi_a(x)) = 0$ we find

$$-\epsilon^2\triangle w - F(a, \phi_a) + F\big(V(\epsilon(x-\xi)), \phi_a + \epsilon^2 w\big) = 0,$$

and dividing by ϵ^2 we form the *rescaled equation*

$$-\triangle w + \epsilon^{-2}\Big[F\big(V(\epsilon(x-\xi)), \phi_a + \epsilon^2 w\big) - F\big(a, \phi_a\big)\Big] = 0, \tag{3.2}$$

which it is convenient to consider in the expanded form

$$\begin{aligned} -\triangle w + \epsilon^{-2}\Big[F\big(V(\epsilon(x-\xi)), \phi_a + \epsilon^2 w\big) - F\big(V(\epsilon(x-\xi)), \phi_a\big)\Big] \\ + \epsilon^{-2}\Big[F\big(V(\epsilon(x-\xi)), \phi_a\big) - F\big(a, \phi_a\big)\Big] = 0. \end{aligned} \tag{3.3}$$

For each $\epsilon > 0$ this defines a C^1 mapping of the variable $(s, w) \in \mathbf{R}^n \times W$ to L^2. This follows from Lemma 3.2 and Lemma 3.6. Bearing in mind that $\xi = -\frac{b}{\epsilon} + s$ and that b is a critical point of V it is plausible (for example by an uncritical application of l'Hopital's rule) that the rescaled equation has a *limit equation* as $\epsilon \to 0^+$, namely,

$$-\triangle w + \frac{\partial F}{\partial u}(a, \phi_a)w + \frac{1}{2}\frac{\partial F}{\partial a}\big(a, \phi_a\big)\big(H(b)(x-s)\cdot(x-s)\big) = 0. \tag{3.4}$$

In order to make a deduction about solutions of (3.3) from solutions of (3.4) we must make sure the limit occurs in the L^2-norm. This is guaranteed by the following considerations. Let us write

$$\begin{aligned} &\epsilon^{-2}\Big[F\big(V(\epsilon(x-\xi)), \phi_a + \epsilon^2 w\big) - F\big(V(\epsilon(x-\xi)), \phi_a\big)\Big] \\ &\qquad = \int_0^1 \frac{\partial F}{\partial u}\Big(V(\epsilon(x-s)+b), \phi_a + \tau\epsilon^2 w\Big) w \, d\tau. \end{aligned}$$

By Lemma 3.3 the integrand converges to $\frac{\partial F}{\partial u}\big(a,\phi_a\big)w$ in L^2 for each τ and by (F3) the L^2-norm of the integrand is bounded by a number independent of τ and ϵ provided the latter is sufficiently small. Hence the integral represents a function converging to $\frac{\partial F}{\partial u}\big(a,\phi_a\big)w$ in L^2.

As for the other term we have, using the fact that $\nabla V(b) = 0$,

$$\begin{aligned}\epsilon^{-2}\Big[F\Big(V\big(\epsilon(x-\xi)\big),\phi_a(x)\Big) - F\Big(a,\phi_a(x)\Big)\Big]\\ = \int_0^1\int_0^1 \frac{\partial F}{\partial a}\Big(V\big(\tau\epsilon(x-s)+b\big),\phi_a\Big)H\Big(\sigma\tau\epsilon(x-s)+b\Big)(x-s)\cdot(x-s)\tau\,d\sigma\,d\tau.\end{aligned}$$

By (F6) the function $\frac{\partial F}{\partial a}\big(V\big(\tau\epsilon(x-s)+b\big),\phi_a\big)$ has uniform exponential decay as $||x|| \to \infty$, uniform, that is, w.r.t. τ and ϵ, and $H(x)$ has polynomial growth. Hence the integrand tends, as a function of x, to $\frac{\partial F}{\partial a}\big(a,\phi_a\big)H(b)(x-s)\cdot(x-s)\tau$ in L^2 and furthermore is, for all $0 < \sigma < 1$, $0 < \tau < 1$ bounded by a fixed function in L^2. Finally we apply Lemma 3.1. This suffices to show that the left-hand side of (3.3) converges to the left-hand side of (3.4) in L^2.

The next step is to solve the limit equation (3.4). This proceeds by means of the Fredholm alternative. We first find s satisfying the n equations

$$\int \frac{\partial F}{\partial a}\Big(a,\phi_a(x)\Big)\big(H(b)(x-s)\cdot(x-s)\big)D_j\phi_a(x)\,dx = 0$$

for $j = 1,\ldots,n$. Since ϕ_a is spherically symmetric, a short calculation reduces this to

$$\left(\frac{2}{n}\int \frac{\partial F}{\partial a}\Big(a,\Phi_a(r)\Big)\Phi'(r)r\,dx\right)\sum_j H_{ij}(b)s_j = 0,$$

and because of (Φ2) this gives

$$\sum_j H_{ij}(b)s_j = 0.$$

Since b is a non-degenerate critical point of V the Hessian matrix is invertible and we obtain a non-degenerate solution $s = 0$. For this solution we have a unique solution $v = \eta \in W$ to the non-homogeneous linear problem (3.1).

We would like to conclude that the rescaled problem has a solution (s_ϵ, w_ϵ) that approaches $(0,\eta)$ but the implicit function theorem requires us to consider the derivative of the rescaled problem. This is the linear map $\mathbf{R}^n \times W \ni (\sigma, v) \mapsto \Gamma_\epsilon(s,w)(\sigma,v) \in L^2$

where (in expanded form, corresponding to (3.3))

$$\Gamma_\epsilon(s,w)(\sigma,v) = -\triangle v + \frac{\partial F}{\partial u}\Big(V(\epsilon(x-s)+b), \phi_a + \epsilon^2 w\Big)v$$

$$- \epsilon^{-1}\left(\frac{\partial F}{\partial a}\Big(V(\epsilon(x-s)+b), \phi_a + \epsilon^2 w\Big) - \frac{\partial F}{\partial a}\Big(V(\epsilon(x-s)+b), \phi_a\Big)\right)\nabla V(\epsilon(x-s)+b)\cdot\sigma$$

$$- \epsilon^{-1}\frac{\partial F}{\partial a}\Big(V(\epsilon(x-s)+b), \phi_a\Big)\nabla V(\epsilon(x-s)+b)\cdot\sigma. \quad (3.5)$$

The problem is apparent in the second term; although $V(\epsilon(x-s)+b)$ converges to a as $\epsilon \to 0$ it does not do so uniformly w.r.t. x. Hence there is no guarantee that the linear map

$$v \mapsto \frac{\partial F}{\partial u}\big(V(\epsilon(x-s)+b), \phi_a + \epsilon^2 w\big)v$$

converges to the linear map $v \mapsto \frac{\partial F}{\partial u}(a,\phi_a)v$ in the operator-norm topology, as would be required by the usual implicit function theorem. On the other hand Lemma 3.3 implies that this operator converges in the strong operator topology and that it depends continuously on ϵ in this topology.

In contrast to this the part of (3.5) representing a linear function of σ presents no topological problems since σ is a finite-dimensional vector. This part converges to

$$-\frac{\partial F}{\partial a}\Big(a,\phi_a\Big)H(b)x\cdot\sigma$$

as $\epsilon \to 0$, $s \to 0$, $w \to \eta$. Let us deal with this first. Taking σ as the ith base vector in $\mathbf{R}^n$ consider first the expression

$$\epsilon^{-1}\left(\frac{\partial F}{\partial a}\Big(V(\epsilon(x-s)+b), \phi_a(x) + \epsilon^2 w\Big) - \frac{\partial F}{\partial a}\Big(V(\epsilon(x-s)+b), \phi_a(x)\Big)\right) D_iV(\epsilon(x-s)+b)$$

$$= \epsilon\int_0^1 \frac{\partial^2 F}{\partial u\partial a}\Big(V(\epsilon(x-s)+b), \phi_a + \tau\epsilon^2 w\Big)wD_iV(\epsilon(x-s)+b)\,d\tau.$$

By (F3) and (V1) the integral remains bounded in L^2 as $\epsilon \to 0$, $s \to 0$ and $w \to \eta$. Hence the expression tends to 0 in L^2.

Second we note that

$$\epsilon^{-1}\frac{\partial F}{\partial a}\Big(V(\epsilon(x-s)+b), \phi_a\Big)D_iV(\epsilon(x-s)+b) \to \frac{\partial F}{\partial a}\big(a,\phi_a\big)(H(b)x)_i$$

in L^2 as $\epsilon \to 0$, $s \to 0$. Indeed the limit holds pointwise (that is, for each x) and

$$\epsilon^{-1}\frac{\partial F}{\partial a}\Big(V(\epsilon(x-s)+b), \phi_a\Big)D_iV(\epsilon(x-s)+b)$$

$$= \frac{\partial F}{\partial a}\Big(V(\epsilon(x-s)+b), \phi_a\Big)\sum_j \int_0^1 D_{i,j}V(t\epsilon(x-s)+b)x_j\,dt$$

and so this function is bounded pointwise, as $\epsilon \to 0$ and $s \to 0$, by a function in L^2, by (F6) and (V1). The conclusion follows by the dominated convergence theorem.

We proceed to verify condition (5) of the modified implicit function theorem. It is in carrying out this verification that condition (P1) comes into play. The object is to show that the linear map $\Gamma_\epsilon(0,\eta)$ is invertible for sufficiently small ϵ and the norm of its inverse has a uniform bound. We apply Lemma 1. First let us note that by Lemma 3.5 the operator $\Gamma_\epsilon(0,\eta)$ is a *compact perturbation* of the linear mapping

$$(\sigma, v) \mapsto -\triangle v + \frac{\partial F}{\partial u}\Big(V(\epsilon x+b), 0\Big)v$$

from $\mathbf{R}^n \times W$ to L^2, which is a Fredholm operator of index 0 thanks to (P1). Hence $\Gamma_\epsilon(0,\eta)$ is a Fredholm operator of index 0. Let $\epsilon_\nu \to 0$ and $(\sigma_\nu, v_\nu) \in \mathbf{R}^n \times W$ be sequences such that

$$||\sigma_\nu|| + ||v_\nu||_{2,2} = 1, \quad \Gamma_{\epsilon_\nu}(0,\eta)(\sigma_\nu, v_\nu) \to 0$$

in the L^2-norm. We obtain a contradiction as follows. Without loss of generality we may assume that $\sigma_\nu \to \sigma$ and $v_\nu \to v$ weakly in H^2. Note that $v \in W$. In view of (3.5) it is plausible that we can proceed to the distribution limit and deduce

$$-\triangle v + \frac{\partial F}{\partial u}\Big(a, \phi_a\Big)v - \frac{\partial F}{\partial a}\Big(a, \phi_a\Big)H(b)x \cdot \sigma = 0. \tag{3.6}$$

We now verify this limit in detail. Although the distribution limit would suffice, we can use Lemma 3.4 to show that the limit holds in a finer topology, the *weak topology on the dual of* H^2. The part linear in σ was dealt with above; it converges in L^2. And $\triangle v_\nu \to \triangle v$ weakly in L^2. Both these topologies are finer than the one in question. It only remains to verify that

$$\frac{\partial F}{\partial u}\Big(V(\epsilon_\nu x+b), \phi_a + \epsilon_\nu^2\eta\Big)v_\nu \to \frac{\partial F}{\partial u}\Big(a, \phi_a\Big)v$$

in the weak topology on the dual of H^2. This follows because, firstly,

$$\frac{\partial F}{\partial u}\Big(a, \phi_a\Big)v_\nu \to \frac{\partial F}{\partial u}\Big(a, \phi_a\Big)v$$

weakly in L^2 (a norm continuous linear map is continuous w.r.t. the weak topologies on domain and codomain); and secondly

$$\left(\frac{\partial F}{\partial u}\Big(V(\epsilon_\nu x+b), \phi_a + \epsilon_\nu^2\eta\Big) - \frac{\partial F}{\partial u}\Big(a, \phi_a\Big)\right)v_\nu \to 0$$

in the weak topology on the dual of H^2 by virtue of Lemma 3.4.

Equation (3.6) being established we draw the conclusion that $\sigma = 0$ (using the Fredholm alternative and the non-singularity of $H(b)$), and hence, since $v \in W$, that $v = 0$. Since $\sigma = 0$ it follows by (3.5) that

$$-\triangle v_\nu + \frac{\partial F}{\partial u}\Big(V(\epsilon_\nu x + b), \phi_a\Big) v_\nu \to 0 \tag{3.7}$$

in the L^2-norm. (Note that (F3) allows us to drop "$\epsilon_\nu^2 \eta$" within the argument of $\frac{\partial F}{\partial u}$ as its contribution tends to 0 in the operator norm.) But

$$\left(\frac{\partial F}{\partial u}\Big(V(\epsilon_\nu x + b), \phi_a\Big) - \frac{\partial F}{\partial u}\Big(V(\epsilon_\nu x + b), 0\Big)\right) v_\nu \to 0 \tag{3.8}$$

in the L^2-norm by Lemma 3.5. So now we have that

$$-\triangle v_\nu + \frac{\partial F}{\partial u}\Big(V(\epsilon_\nu x + b), 0\Big) v_\nu \to 0$$

in the L^2-norm. By conditions (P1) and (V1) there exist K and $\delta > 0$ such that

$$\delta < \frac{\partial F}{\partial u}\Big(V(\epsilon_\nu x + b), 0\Big) < K$$

for all ν. By Wang's Lemma (Lemma 3.7) we deduce the contradiction $v_\nu \to 0$ in the H^2-norm.

It remains to verify condition (6) of the modified implicit function theorem. This requires that $\Gamma_\epsilon(s, w) - \Gamma_\epsilon(0, \eta) \to 0$ as $\epsilon \to 0$, $s \to 0$, $w \to \eta$. As the part of the derivative "linear in σ" converges in norm we have

$$\Gamma_\epsilon(s, w) P_{\mathbf{R}^n} - \Gamma_\epsilon(0, \eta) P_{\mathbf{R}^n} \to 0$$

where $P_{\mathbf{R}^n}$ denotes the projection of $\mathbf{R}^n \times W$ to $\mathbf{R}^n \times 0$. It remains to show that

$$\Gamma_\epsilon(s, w) P_W - \Gamma_\epsilon(0, \eta) P_W \to 0$$

where P_W is the complementary projection. Now

$$\begin{aligned}(\Gamma_\epsilon(s, w) P_W &- \Gamma_\epsilon(0, \eta) P_W) v \\ &= \frac{\partial F}{\partial u}\Big(V(\epsilon(x - s) + b), \phi_a + \epsilon^2 w\Big) v - \frac{\partial F}{\partial u}\Big(V(\epsilon x + b), \phi_a + \epsilon^2 \eta\Big) v.\end{aligned}$$

The conclusion follows from the uniform continuity of $\mathbf{F_u}$ on bounded sets stated in Lemma 3.2.

Finally, we have already noted that $\Gamma_\epsilon(0, \eta)$ is a continuous function of ϵ in the strong operator topology. This implies, by Theorem 2.1, that the solution u_ϵ depends continuously on ϵ. This ends the proof.

4 Multibump solutions

We shall need slightly strengthened versions of some of the conditions of the previous section. The essential difference is the appearance of linear operators with H^1 as a domain instead of H^2.

(F1.1) *In addition to (F1) the derivative* $\dfrac{\partial^3 F}{\partial u^2 \partial a}$ *exists and is continuous.*

(F2.1) *As regards* F, $\dfrac{\partial F}{\partial a}$, $\dfrac{\partial^2 F}{\partial a^2}$ *we make the same assumptions as in (F2). In addition we assume that* $\dfrac{\partial F}{\partial u}$, $\dfrac{\partial^2 F}{\partial u \partial a}$, *define maps*

$$\tilde{\mathbf{F}}_{\mathbf{u}}, \tilde{\mathbf{F}}_{\mathbf{au}} : L^\infty \times H^2 \to \mathcal{L}(H^1, L^2)$$

given by

$$\begin{aligned} \tilde{\mathbf{F}}_{\mathbf{u}}(m,u)v &= \frac{\partial F}{\partial u}\Big(m,u\Big)v, \\ \tilde{\mathbf{F}}_{\mathbf{au}}(m,u)v &= \frac{\partial^2 F}{\partial u \partial a}\Big(m,u\Big)v \end{aligned}$$

and that $\dfrac{\partial^2 F}{\partial u^2}$ *defines a map*

$$\tilde{\mathbf{F}}_{\mathbf{uu}} : L^\infty \times H^2 \to \mathcal{L}_2(H^2 \times H^1, L^2)$$

given by

$$\tilde{\mathbf{F}}_{\mathbf{uu}}(m,u)(v_1,v_2) = \frac{\partial^2 F}{\partial u^2}\Big(m,u\Big)v_1 v_2.$$

(F3.1) *The operators defined in (F2.1) map bounded sets to bounded sets.*

(F4.1) *In addition to (F4) we assume the following. Let* $M \subset L^\infty$ *be bounded,* $u \in H^2$, $v \in H^1$, *and let* h_ν *be a bounded sequence in* L^∞ *converging pointwise to* 0. *Then*

$$\tilde{\mathbf{F}}_{\mathbf{au}}(m,u)vh_\nu \to 0$$

in L^2 *uniformly with respect to* $m \in M$.

(F5.1) *Let* v_ν *be a sequence in* H^2 *with weak limit* 0 *and let* $w \in H^1$. *Then* $\tilde{\mathbf{F}}_{\mathbf{uu}}(m,u)(v_\nu, w) \to 0$ *in* L^2 *uniformly for* (m,u) *in bounded subsets of* $L^\infty \times H^2$.

Lemma 4.1 *The map* $s \mapsto F(m, u(x+s))$ *is* C^1 *from* $\mathbf{R}^n$ *to* L^2.

Proof

$$h^{-1}\Big(F(m,u(x+s+he_i))-F(m,u(x+s))\Big)-\frac{\partial F}{\partial u}\Big(m,u(x+s)\Big)D_iu(x+s)$$
$$=\int_0^1\frac{\partial F}{\partial u}\Big(m,(1-\tau)u(x+s)+\tau u(x+s+he_i)\Big)$$
$$\left(\frac{u(x+s+he_i)-u(x+s)}{h}-D_iu(x+s)\right)d\tau$$
$$+\int_0^1\left[\frac{\partial F}{\partial u}\Big(m,(1-\tau)u(x+s)+\tau u(x+s+he_i)\Big)-\frac{\partial F}{\partial u}\Big(m,u(x+s)\Big)\right]D_iu(x+s)\,d\tau.$$

Since

$$\frac{u(x+s+he_i)-u(x+s)}{h}-D_iu(x+s)\to 0$$

in H^1 the first integral represents a function that tends to 0 in L^2 by (F3.1). The second integral may be written

$$\int\int_S\frac{\partial^2F}{\partial u^2}\Big(m,(1-\sigma\tau)u(x+s)+\sigma\tau u(x+s+he_i)\Big)$$
$$\Big(u(x+s+he_i)-u(x+s)\Big)D_iu(x+s)\tau\,d\sigma\,d\tau$$

and tends to 0 by (F3.1).

We shall need the following addition to Lemma 3.3. The proof, based on (F3.1) and (F4.1) should be obvious.

Lemma 4.2 *Let $m_\nu\in L^\infty$ be a bounded sequence that tends pointwise to $m\in L^\infty$, let $u_\nu\in H^2$ converge to $u\in H^2$ and let $v\in H^1$. Then*

$$\tilde{\mathbf{F}}_{\mathbf{u}}(m_\nu,u_\nu)v\to\tilde{\mathbf{F}}_{\mathbf{u}}(m,u)v,$$

in other words $\tilde{\mathbf{F}}_{\mathbf{u}}(m_\nu,u_\nu)\to\tilde{\mathbf{F}}_{\mathbf{u}}(m,u)$ in the strong operator topology.

Theorem 4.3 *Make the same assumptions on F, V and ϕ_a as in Theorem 3.8 except that (F1.1), (F2.1), (F3.1), (F4.1) and (F5.1) replace (F1), (F2), (F3), (F4) and (F5) respectively. Let b_1 and b_2 be distinct non-degenerate critical points of V, let $a_k=V(b_k)$, $\phi^{(k)}=\phi_{a_k}$, $k=1,2$. Then for all sufficiently small $\epsilon>0$ equation (1.2) has a solution u_ϵ of the form*

$$u_\epsilon(x)=\sum_{k=1}^{2}\phi^{(k)}\Big(x-\frac{b_k}{\epsilon}+s_\epsilon^{(k)}\Big)+\epsilon^2w_\epsilon^{(k)}\Big(x-\frac{b_k}{\epsilon}+s_\epsilon^{(k)}\Big)$$

such that $s_\epsilon^{(k)} \in \mathbf{R}^n$, $w_\epsilon^{(k)} \in H^2$ *and is* L^2*-orthogonal to the functions* $D_j\phi^{(k)}$ *for* $j = 1,\dots,n$, *and* $s_\epsilon^{(k)}$ *and* $w_\epsilon^{(k)}$ *depend continuously on* ϵ. *Moreover* $s_\epsilon^{(k)} \to 0$ *and* $w_\epsilon^{(k)} \to \eta^{(k)}$ *in the* H^2*-norm as* $\epsilon \to 0^+$ *where* $\eta^{(k)}$ *is the unique solution orthogonal to the n functions* $D_j\phi^{(k)}(x)$ *of the non-homogeneous linear problem*

$$-\triangle v + \frac{\partial F}{\partial u}\Big(a_k, \phi^{(k)}(x)\Big)v = -\frac{1}{2}\frac{\partial F}{\partial a}\Big(a_k, \phi^{(k)}\Big)\big(H(b_k)x\cdot x\big) \tag{4.1}$$

and $H(x)$ *is the Hessian matrix* $D_{i,j}V(x)$.

Proof Let

$$M(a,u_1,u_2) = F(a,u_1+u_2) - F(a,u_1) - F(a,u_2).$$

Consider the pair of equations

$$\begin{array}{rcl} -\triangle u_1 + F(V(\epsilon x),u_1) + \lambda_1 M(V(\epsilon x),u_1,u_2) &=& 0 \\ -\triangle u_2 + F(V(\epsilon x),u_2) + \lambda_2 M(V(\epsilon x),u_1,u_2) &=& 0 \end{array} \tag{4.2}$$

where $\lambda_1 + \lambda_2 = 1$. If u_1, u_2 satisfy (4.2) then $u_1 + u_2$ satisfies (1.2). The idea is to use scaling to find a solution of (4.2) with u_k near to $\phi^{(k)}(x - \frac{b_k}{\epsilon})$.

For $k = 1,2$ define

$$W_k = \Big\{ w \in H^2 : \int wD_j\phi^{(k)} = 0, j = 1,\dots,n \Big\}.$$

Introduce new state variables $s^{(k)} \in \mathbf{R}^n$, $w^{(k)} \in W_k$ for $k = 1,2$, where

$$u_k(x) = \phi^{(k)}\Big(x - \frac{b_k}{\epsilon} + s^{(k)}\Big) + \epsilon^2 w^{(k)}\Big(x - \frac{b_k}{\epsilon} + s^{(k)}\Big).$$

We shall often denote the pair $(s^{(1)}, s^{(2)})$ by s and the pair $(w^{(1)}, w^{(2)})$ by w. Substitute into (4.2) and proceed as in the proof of Theorem 3.8. We shall make use of some abbreviations in order to fit the equations on to the line. We write

$$\xi^{(k)} = -\frac{b_k}{\epsilon} + s^{(k)}, \quad \delta^{(ij)} = \xi^{(j)} - \xi^{(i)}.$$

We replace x by $x - \xi^{(1)}$ in the first equation and by $x - \xi^{(2)}$ in the second. This produces the rescaled equations

$$\begin{aligned} -\triangle w^{(1)} &+ \epsilon^{-2}\Big(F\big(V(\epsilon(x-\xi^{(1)})), \phi^{(1)} + \epsilon^2 w^{(1)}\big) - F\big(a_1, \phi^{(1)}\big)\Big) \\ &+ \lambda_1 \epsilon^{-2} M\Big(V(\epsilon(x-\xi^{(1)})), \phi^{(1)} + \epsilon^2 w^{(1)}, (\phi^{(2)} + \epsilon^2 w^{(2)})(x + \delta^{(12)})\Big) = 0 \end{aligned} \tag{4.3}$$

and

$$
\begin{aligned}
-\triangle w^{(2)}+\epsilon^{-2}\Big(F\big(V(\epsilon(x-\xi^{(2)})),\phi^{(2)}+\epsilon^2 w^{(2)}\big)-F\big(a_2,\phi^{(2)}\big)\Big)&\\
+\lambda_2\epsilon^{-2}M\Big(V(\epsilon(x-\xi^{(2)})),(\phi^{(1)}+\epsilon^2 w^{(1)})(x+\delta^{(21)}),\phi^{(2)}+\epsilon^2 w^{(2)}\Big)&=0. \quad (4.4)
\end{aligned}
$$

Note that Lemma 4.1 is needed to show that the left-hand sides of the rescaled equations constitute a C^1 function of (s,w).

As $\epsilon\to 0$ we claim that the terms coupling these two equations, that is, the terms involving the function M, tend to 0 in L^2, and the limit problem is the uncoupled pair

$$
-\triangle w^{(k)}+\frac{\partial F}{\partial u}\big(a_k,\phi^{(k)}\big)w^{(k)}+\frac{1}{2}\frac{\partial F}{\partial a}\big(a_k,\phi^{(k)}\big)\big(H(b_k)(x-s^{(k)})\cdot(x-s^{(k)})\big)=0 \quad (4.5)
$$

for $k=1,2$, with the non-degenerate solution $s=0$, $w=\eta=(\eta^{(1)},\eta^{(2)})$. The required conclusions follow as in Theorem 3.8 provided we can justify the use of the Modified Implicit Function Theorem. This requires a careful examination of the coupling terms in (4.3, 4.4).

First we consider the L^2-limit of the coupling terms evaluated at the solution of the limit problem $s^{(k)}=0$, $w^{(k)}=\eta^{(k)}$. By elementary calculus

$$
M(a,u_1,u_2)=\int_0^1\int_0^1\frac{\partial^2 F}{\partial u^2}\Big(a,\sigma u_1+\tau u_2\Big)u_1u_2\,d\sigma\,d\tau.
$$

So we may drop the integral provided all limits occur uniformly w.r.t. σ,τ in the unit square. Corresponding to the coupling term in (4.3) we obtain for consideration

$$
\begin{aligned}
\epsilon^{-2}\frac{\partial^2 F}{\partial u^2}\Big(V(\epsilon(x-\xi^{(1)})),\sigma\big(\phi^{(1)}+\epsilon^2\eta^{(1)}\big)+\tau\big(\phi^{(2)}+\epsilon^2\eta^{(2)}\big)(x+\delta^{(12)})\Big)&\\
\Big(\phi^{(1)}+\epsilon^2\eta^{(1)}\Big)\Big(\phi^{(2)}(x+\delta^{(12)})+\epsilon^2\eta^{(2)}(x+\delta^{(12)})\Big)&
\end{aligned}
$$

where

$$
\delta^{(12)}=\xi^{(2)}-\xi^{(1)}=\frac{b_1-b_2}{\epsilon}+s^{(2)}-s^{(1)}
$$

whence, since b_1 and b_2 are distinct, $||\delta^{(12)}||\to\infty$ as $\epsilon\to 0^+$. In view of the boundedness property (F3) we can distinguish three terms that need to be examined after multiplying out this expression; the rest is $O(\epsilon^2)$. The first is

$$
\begin{aligned}
\epsilon^{-2}\frac{\partial^2 F}{\partial u^2}\Big(V(\epsilon(x-\xi^{(1)})),\sigma\big(\phi^{(1)}+\epsilon^2\eta^{(1)}\big)+\tau\big(\phi^{(2)}+\epsilon^2\eta^{(2)}\big)(x+\delta^{(12)})\Big)&\\
\phi^{(1)}\phi^{(2)}(x+\delta^{(12)}) \quad (4.6)&
\end{aligned}
$$

To estimate this we write $\phi^{(1)} = \chi\psi$ where both functions have exponential decay and χ is smooth (for example take ρ as a smooth function such that $\rho(x) = ||x||$ for $||x|| > 1$ and set $\chi(x) = e^{-k\rho(x)}$ where k is suitably small and positive). Then the L^2-norm of the function represented in (4.6) is bounded according to (F3) by a constant times

$$\epsilon^{-2} \|\chi\|_{2,2} \cdot \left\| \psi\phi^{(2)}(\cdot + \delta^{(12)}) \right\|_{2,2}.$$

This tends to 0; in fact it tends to 0 faster than any power of ϵ since owing to the exponential decay of $\phi^{(k)}$ and its derivatives there are constants A and $B > 0$ such that

$$\left\| \psi\phi^{(2)}(\cdot + \delta^{(12)}) \right\|_{2,2} \le Ae^{-B/\epsilon}.$$

In passing we can mention that this step is easier to carry out if we assume growth conditions on F instead of the boundedness property of (F3).

The second term is

$$\frac{\partial^2 F}{\partial u^2}\left(V(\epsilon(x-\xi^{(1)})), \sigma\big(\phi^{(1)} + \epsilon^2\eta^{(1)}\big) + \tau\big(\phi^{(2)} + \epsilon^2\eta^{(2)}\big)(x+\delta^{(12)}) \right)\eta^{(1)}\phi^{(2)}(x+\delta^{(12)}).$$

Since $\phi^{(2)}(x + \delta^{(12)}) \to 0$ weakly in H^2 this tends to 0 uniformly for $0 < \sigma, \tau < 1$ by (F5.1).

The third term is

$$\frac{\partial^2 F}{\partial u^2}\left(V(\epsilon(x-\xi^{(1)})), \sigma\big(\phi^{(1)} + \epsilon^2\eta^{(1)}\big) + \tau\big(\phi^{(2)} + \epsilon^2\eta^{(2)}\big)(x+\delta^{(12)}) \right)\phi^{(1)}\eta^{(2)}(x+\delta^{(12)}).$$

This tends to 0 uniformly for $0 < \sigma, \tau < 1$, as may be seen by (F5.1) after replacing x by $x + \delta^{(21)}$.

Next we consider the derivative of the rescaled problem with special attention to the coupling terms as the rest was effectively dealt with in the previous section. The first objective is to show that the derivative w.r.t. s of the coupling terms tends to 0 in norm. We consider the partial derivatives of the *coupling terms* of (4.3, 4.4) w.r.t. the coordinates of $s^{(1)}$, $s^{(2)}$ (note that we rely on Lemmas 3.6 and 4.1 here for their existence) with a view to showing that they tend to 0 in L^2 as $\epsilon \to 0$, $s \to 0$, $w \to \eta$. Since the partial derivatives of $\xi^{(k)}$ and $\delta^{(ij)}$ w.r.t. the coordinates of $s^{(1)}$, $s^{(2)}$ are constants, it is convenient to view the rescaled problem as a function of $\xi^{(k)}$ and $\delta^{(ij)}$, and show that the partial derivatives w.r.t. the coordinates of these vectors tend to 0 in L^2.

The partial derivative of the coupling term of (4.3) w.r.t. the ith coordinate of $\xi^{(1)}$ is the function

$$
\begin{aligned}
-\lambda_1\epsilon^{-1}\Bigg[\frac{\partial F}{\partial a}\Big(V(\epsilon(x-\xi^{(1)})),\phi^{(1)}+\epsilon^2 w^{(1)}+(\phi^{(2)}+\epsilon^2 w^{(2)})(x+\delta^{(12)})\Big) \\
-\frac{\partial F}{\partial a}\Big(V(\epsilon(x-\xi^{(1)})),\phi^{(1)}+\epsilon^2 w^{(1)}\Big) \\
-\frac{\partial F}{\partial a}\Big(V(\epsilon(x-\xi^{(1)})),(\phi^{(2)}+\epsilon^2 w^{(2)})(x+\delta^{(12)})\Big)\Bigg] D_i V(\epsilon(x-\xi^{(1)})) \quad (4.7)
\end{aligned}
$$

where $\delta^{(12)}=\frac{b_1-b_2}{\epsilon}+s^{(2)}-s^{(1)}$. By the now usual argument involving (F3.1) we can drop the inner terms involving ϵ^2. This leaves

$$
\begin{aligned}
&-\lambda_1\epsilon^{-1}\Bigg[\frac{\partial F}{\partial a}\Big(V(\epsilon(x-\xi^{(1)})),\phi^{(1)}+\phi^{(2)}(x+\delta^{(12)})\Big)-\frac{\partial F}{\partial a}\Big(V(\epsilon(x-\xi^{(1)})),\phi^{(1)}\Big) \\
&\qquad -\frac{\partial F}{\partial a}\Big(V(\epsilon(x-\xi^{(1)})),\phi^{(2)}(x+\delta^{(12)})\Big)\Bigg] D_i V(\epsilon(x-\xi^{(1)})) \\
&= -\lambda_1\epsilon^{-1} D_i V(\epsilon(x-\xi^{(1)}))\iint_S \frac{\partial^3 F}{\partial^2 u\partial a}\Big(V(\epsilon(x-\xi^{(1)})),\sigma\phi^{(1)}+\tau\phi^{(2)}(x+\delta^{(12)})\Big) \\
&\qquad \phi^{(1)}\phi^{(2)}(x+\delta^{(12)})\,d\sigma\,d\tau.
\end{aligned}
$$

Since $\phi^{(1)}$, $\phi^{(2)}$ and V are bounded functions the absolute value of this function is bounded by a constant times

$$
|\epsilon^{-1}D_i V(\epsilon(x-\xi^{(1)}))|.|\phi^{(1)}|.|\phi^{(2)}(x+\delta^{(12)})|,
$$

a function that tends to 0 in L^2 as $\epsilon\to 0$, $s\to 0$ by the dominated convergence theorem. Note that we only use the continuity of $\partial^3 F/\partial^2 u\,\partial a$, we don't need to know any property of the operator it defines.

The partial derivative of the coupling term of (4.3) w.r.t. the ith coordinate of $\delta^{(12)}$ is the function

$$
\begin{aligned}
\lambda_1\epsilon^{-2}\Bigg[\frac{\partial F}{\partial u}\Big(V(\epsilon(x-\xi^{(1)})),\phi^{(1)}+\epsilon^2 w^{(1)}+(\phi^{(2)}+\epsilon^2 w^{(2)})(x+\delta^{(12)})\Big) \\
-\frac{\partial F}{\partial u}\Big(V(\epsilon(x-\xi^{(1)})),(\phi^{(2)}+\epsilon^2 w^{(2)})(x+\delta^{(12)})\Big)\Bigg] \\
\Big(D_i\phi^{(2)}(x+\delta^{(12)})+\epsilon^2 D_i w^{(2)}(x+\delta^{(12)})\Big). \quad (4.8)
\end{aligned}
$$

By the usual device we consider

$$\epsilon^{-2}\frac{\partial^2 F}{\partial u^2}\Big(V(\epsilon(x-\xi^{(1)})),\tau(\phi^{(1)}+\epsilon^2 w^{(1)})+(\phi^{(2)}+\epsilon^2 w^{(2)})(x+\delta^{(12)})\Big)$$
$$(\phi^{(1)}+\epsilon^2 w^{(1)})\Big(D_i\phi^{(2)}(x+\delta^{(12)})+\epsilon^2 D_i w^{(2)}(x+\delta^{(12)})\Big).$$

After multiplying out the product we have a term $O(\epsilon^2)$ and three others. These are, firstly

$$\epsilon^{-2}\frac{\partial^2 F}{\partial u^2}\Big(V(\epsilon(x-\xi^{(1)})),\tau(\phi^{(1)}+\epsilon^2 w^{(1)})+(\phi^{(2)}+\epsilon^2 w^{(2)})(x+\delta^{(12)})\Big)$$
$$\phi^{(1)}D_i\phi^{(2)}(x+\delta^{(12)}),$$

secondly

$$\frac{\partial^2 F}{\partial u^2}\Big(V(\epsilon(x-\xi^{(1)})),\tau(\phi^{(1)}+\epsilon^2 w^{(1)})+(\phi^{(2)}+\epsilon^2 w^{(2)})(x+\delta^{(12)})\Big)$$
$$w^{(1)}D_i\phi^{(2)}(x+\delta^{(12)})$$

and thirdly

$$\frac{\partial^2 F}{\partial u^2}\Big(V(\epsilon(x-\xi^{(1)})),\tau(\phi^{(1)}+\epsilon^2 w^{(1)})+(\phi^{(2)}+\epsilon^2 w^{(2)})(x+\delta^{(12)})\Big)$$
$$\phi^{(1)}D_i w^{(2)}(x+\delta^{(12)}).$$

These tend to 0 in L^2 as $\epsilon\to 0$, $s\to 0$, $w\to\eta$, uniformly for $0<\tau<1$ by familiar arguments involving (F5.1) and the exponential decay of ϕ and its first derivatives. (For the third one replace x by $x+\delta^{(21)}$.) The second equation of the rescaled problem is dealt with similarly.

Next we consider the total derivative of the left-hand side of (4.3) at the solution of the limit equation $w=\eta$, $s=0$. This is the linear map $\Gamma_\epsilon^{(1)}(0,\eta)$ from $\mathbf{R}^n\times\mathbf{R}^n\times W_1\times W_2$ to L^2 given by

$$\begin{aligned}\Gamma_\epsilon^{(1)}(0,\eta)(\sigma,v) &= -\triangle v^{(1)}+(1-\lambda_1)\frac{\partial F}{\partial u}\Big(V(\epsilon(x-\xi^{(1)})),\phi^{(1)}+\epsilon^2\eta^{(1)}\Big)v^{(1)}\\
&+\lambda_1\frac{\partial F}{\partial u}\Big(V(\epsilon(x-\xi^{(1)})),\phi^{(1)}+\epsilon^2\eta^{(1)}+(\phi^{(2)}+\epsilon^2\eta^{(2)})(x+\delta^{(12)})\Big)\big(v^{(1)}+v^{(2)}(x+\delta^{(12)})\big)\\
&-\lambda_1\frac{\partial F}{\partial u}\Big(V(\epsilon(x-\xi^{(1)})),(\phi^{(2)}+\epsilon^2\eta^{(2)})(x+\delta^{(12)})\Big)v^{(2)}(x+\delta^{(12)})\\
&+\Gamma_\epsilon^{(1)}(0,\eta)(\sigma,0),\end{aligned}\tag{4.9}$$

where the final term, giving the part that is a linear function of σ, is known by the above analysis, and the proof of Theorem 3.8, to converge in L^2 to

$$-\frac{\partial F}{\partial a}\Big(a_1,\phi^{(1)}\Big)H(b_1)x\cdot\sigma^{(1)}.$$

A similar formula will hold for the second component $\Gamma_\epsilon^{(2)}(0,\eta)$. We note that by Lemma 3.5 the operator $W_1\times W_2\ni v\mapsto\Gamma_\epsilon(0,\eta)(0,v)\in L^2\times L^2$ is, for each ϵ, a compact perturbation of the operator

$$v\mapsto\Bigg(-\triangle v^{(1)}+\frac{\partial F}{\partial u}\Big(V(\epsilon(x-\xi^{(1)})),0\Big)v^{(1)},-\triangle v^{(2)}+\frac{\partial F}{\partial u}\Big(V(\epsilon(x-\xi^{(1)})),0\Big)v^{(2)}\Bigg)$$

which, thanks to (P1), is Fredholm operator of index $-2n$. Hence $\Gamma_\epsilon(0,\eta)$ is a Fredholm operator of index 0.

We therefore verify condition (5) of the modified implicit function theorem by assuming that there exist sequences

$$\epsilon_\nu\to 0,\quad \sigma_\nu\in\mathbf{R}^n\times\mathbf{R}^n,\quad v_\nu\in W_1\times W_2$$

such that

$$||\sigma_\nu||+||v_\nu||_{2,2}=1$$

whilst

$$\Gamma_{\epsilon_\nu}(0,\eta)(\sigma_\nu,v_\nu)\to 0$$

in L^2, and deduce a contradiction. We write ξ_ν, δ_ν to denote the corresponding quantities when ϵ_ν replaces ϵ.

Without loss of generality we may assume that $\sigma_\nu\to\sigma$ and the limits

$$v_\nu^{(1)}\to y^{(1)},\quad v_\nu^{(2)}\to y^{(2)},\quad v_\nu^{(1)}(\cdot+\delta_\nu^{(21)})\to z^{(1)},\quad v_\nu^{(2)}(\cdot+\delta_\nu^{(12)})\to z^{(2)},$$

hold in the weak topology on H^2. Note that $y^{(k)}\in W_k$. Referring now to formula (4.9) and applying Lemma 3.4 we compute the limit of $\Gamma_{\epsilon_\nu}^{(1)}(0,\eta)(\sigma_\nu,v_\nu)$ in the weak topology on the dual of H^2, (compare the corresponding part of the proof of Theorem 3.8; again the distribution limit would suffice), and deduce that

$$\begin{aligned}-\triangle y^{(1)}+\frac{\partial F}{\partial u}\Big(a_1,\phi^{(1)}\Big)y^{(1)}+\lambda_1\frac{\partial F}{\partial u}\Big(a_1,\phi^{(1)}\Big)z^{(2)}&\\ -\lambda_1\frac{\partial F}{\partial u}\Big(a_1,0\Big)z^{(2)}-\frac{\partial F}{\partial a}\Big(a_1,\phi^{(1)}\Big)H(b_1)x\cdot\sigma^{(1)}&=0.\end{aligned}\tag{4.10}$$

Furthermore, if we replace x by $x+\delta^{(21)}$ in formula (4.9), we can again take the weak limit in the dual of H^2 and deduce

$$-\triangle z^{(1)}+(1-\lambda_1)\frac{\partial F}{\partial u}\Big(a_2,0\Big)z^{(1)}+\lambda_1\frac{\partial F}{\partial u}\Big(a_2,\phi^{(2)}\Big)z^{(1)}=0.\tag{4.11}$$

In a similar way we deduce that

$$
\begin{aligned}
&-\triangle y^{(2)} + \frac{\partial F}{\partial u}\Big(a_2, \phi^{(2)}\Big) y^{(2)} + \lambda_2 \frac{\partial F}{\partial u}\Big(a_2, \phi^{(2)}\Big) z^{(1)} \\
&\qquad\qquad - \lambda_2 \frac{\partial F}{\partial u}\Big(a_2, 0\Big) z^{(1)} - \frac{\partial F}{\partial a}\Big(a_2, \phi^{(2)}\Big) H(b_2) x \cdot \sigma^{(2)} = 0 \qquad (4.12)
\end{aligned}
$$

and

$$
-\triangle z^{(2)} + (1 - \lambda_2) \frac{\partial F}{\partial u}\Big(a_1, 0\Big) z^{(2)} + \lambda_2 \frac{\partial F}{\partial u}\Big(a_1, \phi^{(1)}\Big) z^{(2)} = 0. \qquad (4.13)
$$

Up to this point λ_1 and λ_2 have been arbitrary save that $\lambda_1 + \lambda_2 = 1$. Subject to this condition we now choose λ_1 and λ_2 so that the operators

$$
-\triangle + (1 - \lambda_1) \frac{\partial F}{\partial u}\Big(a_2, 0\Big) + \lambda_1 \frac{\partial F}{\partial u}\Big(a_2, \phi^{(2)}\Big)
$$

and

$$
-\triangle + (1 - \lambda_2) \frac{\partial F}{\partial u}\Big(a_1, 0\Big) + \lambda_2 \frac{\partial F}{\partial u}\Big(a_1, \phi^{(1)}\Big)
$$

are invertible from H^2 to L^2. In fact each is a Fredholm operator of index 0 for all λ_1 and λ_2 owing to Lemma 3.5 and (P1). The first is invertible for $\lambda_1 = 0$, the second for $\lambda_2 = 0$. Each is an analytic function of λ_1, λ_2. Each is therefore invertible for all but a discrete set of values of λ_1 (the first), λ_2 (the second). Hence there exist λ_1 and λ_2 such that both operators are invertible and $\lambda_1 + \lambda_2 = 1$.

For this choice of λ_1 and λ_2 we deduce from (4.11) and (4.13) that $z^{(1)} = z^{(2)} = 0$. Then (4.10) and (4.12) reduce to

$$
-\triangle y^{(1)} + \frac{\partial F}{\partial u}\Big(a_1, \phi^{(1)}\Big) y^{(1)} - \frac{\partial F}{\partial a}\Big(a_1, \phi^{(1)}\Big) H(b_1) x \cdot \sigma^{(1)} = 0
$$

and

$$
-\triangle y^{(2)} + \frac{\partial F}{\partial u}\Big(a_2, \phi^{(2)}\Big) y^{(2)} - \frac{\partial F}{\partial a}\Big(a_2, \phi^{(2)}\Big) H(b_2) x \cdot \sigma^{(2)} = 0.
$$

As in the proof of Theorem 3.8 this leads to $y^{(1)} = y^{(2)} = 0$, but, what is more, $\sigma = (0, 0)$.

Returning to (4.9) we drop the terms with ϵ^2, as their contribution tends to 0 in the L^2-norm by (F3.1). We see that

$$
\begin{aligned}
&-\triangle v_\nu^{(1)} + \frac{\partial F}{\partial u}\Big(V(\epsilon_\nu x + b_1), \phi^{(1)}\Big) v_\nu^{(1)} \\
&\quad + \lambda_1 \left[\frac{\partial F}{\partial u}\Big(V(\epsilon_\nu x + b_1), \phi^{(1)} + \phi^{(2)}(x + \delta_\nu^{(12)})\Big) - \frac{\partial F}{\partial u}\Big(V(\epsilon_\nu x + b_1), \phi^{(1)}\Big) \right] v_\nu^{(1)} \\
&+\lambda_1 \left[\frac{\partial F}{\partial u}\Big(V(\epsilon_\nu x + b_1), \phi^{(1)} + \phi^{(2)}(x + \delta_\nu^{(12)})\Big) - \frac{\partial F}{\partial u}\Big(V(\epsilon_\nu x + b_1), \phi^{(2)}(x + \delta_\nu^{(12)})\Big) \right] v_\nu^{(2)}(x + \delta_\nu^{(12)}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \to 0
\end{aligned}
$$

in L^2. The function spread over the third line tends to 0 in L^2 by Lemma 3.5. The one on the second line also tends to 0 in L^2 since we may apply Lemma 3.5 after replacing x by $x+\delta_\nu^{(21)}$. Hence we find that

$$-\triangle v_\nu^{(1)} + \frac{\partial F}{\partial u}\Big(V(\epsilon_\nu x + b_1), \phi^{(1)}\Big) v_\nu^{(1)} \to 0$$

in L^2. Again by Lemma 3.5

$$\left[\frac{\partial F}{\partial u}\Big(V(\epsilon_\nu x + b_1), \phi^{(1)}\Big) - \frac{\partial F}{\partial u}\Big(V(\epsilon_\nu x + b_1), 0\Big)\right] v_\nu^{(1)} \to 0$$

in L^2 so that we deduce

$$-\triangle v_\nu^{(1)} + \frac{\partial F}{\partial u}\Big(V(\epsilon_\nu x + b_1), 0\Big) v_\nu^{(1)} \to 0$$

in L^2. But now we obtain $||v_\nu^{(1)}||_{2,2} \to 0$ by Wang's Lemma. Similarly $||v_\nu^{(2)}||_{2,2} \to 0$. Since also $\sigma_\nu \to (0,0)$ we obtain $||\sigma_\nu|| + ||v_\nu||_{2,2} \to 0$ in contradiction with the assumption that $||\sigma_\nu|| + ||v_\nu||_{2,2} = 1$. This proves condition (5).

The final step is to verify condition (6). The first component of the derivative is

$$\begin{aligned}\Gamma_\epsilon^{(1)}(s,w)(\sigma,v) &= -\triangle v^{(1)} + (1-\lambda_1)\frac{\partial F}{\partial u}\Big(V(\epsilon(x-\xi^{(1)})), \phi^{(1)} + \epsilon^2 w^{(1)}\Big) v^{(1)} \\ &+ \lambda_1 \frac{\partial F}{\partial u}\Big(V(\epsilon(x-\xi^{(1)})), \phi^{(1)} + \epsilon^2 w^{(1)} + (\phi^{(2)} + \epsilon^2 w^{(2)})(x+\delta^{(12)})\Big)(v^{(1)} + v^{(2)}(x+\delta^{(12)})) \\ &- \lambda_1 \frac{\partial F}{\partial u}\Big(V(\epsilon(x-\xi^{(1)})), (\phi^{(2)} + \epsilon^2 w^{(2)})(x+\delta^{(12)})\Big) v^{(2)}(x+\delta^{(12)}) \\ &+ \Gamma_\epsilon^{(1)}(s,w)(\sigma,0). \qquad (4.14)\end{aligned}$$

Cast out those parts that we know converge in norm as $\epsilon \to 0$, $s \to 0$, $w \to \eta$. These include the part linear in σ and the contribution of the ϵ^2 terms within the arguments of $\frac{\partial F}{\partial u}$. Break the rest into four bits

$$\frac{\partial F}{\partial u}\Big(V(\epsilon(x-\xi^{(1)})), \phi^{(1)}\Big) v^{(1)},$$

$$\frac{\partial F}{\partial u}\Big(V(\epsilon(x-\xi^{(1)})), \phi^{(1)} + \phi^{(2)}(x+\delta^{(12)})\Big) v^{(1)},$$

$$\frac{\partial F}{\partial u}\Big(V(\epsilon(x-\xi^{(1)})), \phi^{(1)} + \phi^{(2)}(x+\delta^{(12)})\Big) v^{(2)}(x+\delta^{(12)}))$$

and

$$\frac{\partial F}{\partial u}\Big(V(\epsilon(x-\xi^{(1)})), \phi^{(2)}(x+\delta^{(12)})\Big) v^{(2)}(x+\delta^{(12)}).$$

Recall that we wish to prove that $\Gamma_\epsilon(s,w)-\Gamma_\epsilon(0,\eta)\to 0$ and note that w doesn't occur in these remaining quantities. Bits 1 and 2 are handled as in Theorem 3.8 using the uniform continuity of $\mathbf{F_u}$ on bounded sets. The others require more care. Using the uniform continuity of $\mathbf{F_u}$ we reduce it to showing that the norms of the operators

$$v\mapsto \frac{\partial F}{\partial u}\Big(V(\epsilon x+b_1),\phi^{(1)}+\phi^{(2)}(x+\frac{b_1-b_2}{\epsilon})\Big)\Big(v(x+\frac{b_1-b_2}{\epsilon}+s^{(2)}-s^{(1)})-v(x+\frac{b_1-b_2}{\epsilon})\Big)$$

and

$$v\mapsto \frac{\partial F}{\partial u}\Big(V(\epsilon x+b_1),\phi^{(2)}(x+\frac{b_1-b_2}{\epsilon})\Big)\Big(v(x+\frac{b_1-b_2}{\epsilon}+s^{(2)}-s^{(1)})-v(x+\frac{b_1-b_2}{\epsilon})\Big)$$

from H^2 to L^2 converge to 0 as $\epsilon\to 0$, $s\to 0$. Replace x by $x+\frac{b_2-b_1}{\epsilon}$ for convenience (it makes no difference to the norm of the operator). We have here the operators (and note the tilde and the placeholder dot for the variable x)

$$\tilde{\mathbf{F}}_{\mathbf{u}}\Big(V(\epsilon(\cdot)+b_2),\phi^{(1)}(\cdot+\frac{b_2-b_1}{\epsilon})+\phi^{(2)}\Big)J_{H^2,H^1}(T_{s^{(2)}-s^{(1)}}-I_{H^2})$$

$$\tilde{\mathbf{F}}_{\mathbf{u}}\Big(V(\epsilon(\cdot)+b_2),\phi^{(1)}\Big)J_{H^2,H^1}(T_{s^{(2)}-s^{(1)}}-I_{H^2})$$

where $T_r: H^2\to H^2$ denotes the translation operator $u\mapsto u(\cdot+r)$, $I_{H^2}: H^2\to H^2$ the identity operator and $J_{H^2,H^1}: H^2\to H^1$ the natural embedding. Thanks to (F3.1) these are bounded in norm by a constant times

$$||J_{H^2,H^1}(T_{s^{(2)}-s^{(1)}}-I_{H^2})||$$

which tends to 0 as $s\to 0$. NB: The difference between (F3) and (F3.1) is essential; $||T_{s^{(2)}-s^{(1)}}-I_{H^2}||$ does not tend to 0.

Finally we note that the map $\epsilon\mapsto\Gamma_\epsilon(0,\eta)$ is continuous in the strong operator topology for $\epsilon>0$, that is, the map $\epsilon\mapsto\Gamma_\epsilon(0,\eta)(\sigma,v)$ is continuous from $\mathbf{R}$ to L^2 on the domain $\epsilon>0$ for each (σ,v). This follows from Lemmas 3.3 and 4.2 in conjunction with formulas (4.7, 4.8, 4.9). We conclude that u_ϵ depends continuously on ϵ. This ends the proof.

5 Extension to m bumps

The extension of the arguments of the preceding section to the case of m bumps is mostly a question of devising a readable notation in which to express the somewhat lengthy equations. Let us suppose that we have non-degenerate critical points b_k, $k=1,\dots,m$ with values $a_k=V(b_k)$. Let $\phi^{(k)}=\phi_{a_k}$. Corresponding to the equation pair (4.2) we have the system

$$-\triangle u_k+F(V(\epsilon x),u_k)+\lambda_k M(V(\epsilon x),u_1,\dots,u_m)=0,\quad k=1\dots,m \tag{5.1}$$

where

$$M(a, u_1, \ldots, u_m) = F\Big(a, \sum_{k=1}^{m} u_k\Big) - \sum_{k=1}^{m} F(a, u_k)$$

and the constants λ_k are chosen so that $\sum_{k=1}^{m} \lambda_k = 1$ and satisfy another condition to be specified later. The rescaled variables are $s^{(k)} \in \mathbf{R}^n$, $w^{(k)} \in W_k$, defined as in the previous section. The rescaled equations can be expressed in the form

$$\begin{aligned}
&- \triangle w^{(k)} + \epsilon^{-2}\Big(F\big(V(\epsilon(x-\xi^{(k)})), \phi^{(k)} + \epsilon^2 w^{(k)}\big) - F\big(a_k, \phi^{(k)}\big)\Big) \\
&+ \lambda_k \epsilon^{-2} M\Big(V(\epsilon(x-\xi^{(k)})), \big(\phi^{(1)} + \epsilon^2 w^{(1)}\big)(x+\delta^{(k1)}), \ldots, \big(\phi^{(m)} + \epsilon^2 w^{(m)}\big)(x+\delta^{(km)})\Big) \\
&= 0 \quad (5.2)
\end{aligned}$$

where $\xi^{(k)} = -\frac{b_k}{\epsilon} + s^{(k)}$, $\delta^{(ij)} = \xi^{(j)} - \xi^{(i)}$. To prove that the coupling terms tend to 0 in L^2 as $\epsilon \to 0^+$ we use the expansion

$$\begin{aligned}
M(a, u_1, \ldots, u_m) &= F\Big(a, \sum_{i=1}^{m} u_i\Big) - \sum_{i=1}^{m} F(a, u_i) \\
&= \sum_{j=1}^{m} \Big(F\Big(a, \sum_{i=j}^{m} u_i\Big) - F(a, u_j) - F\Big(a, \sum_{i=j+1}^{m} u_i\Big)\Big) \\
&= \sum_{j=1}^{m} J_j(a, u_1, \ldots, u_m) u_j \sum_{i=j+1}^{m} u_i \\
&= \sum_{1 \le j < i \le m} J_j(a, u_1, \ldots, u_m) u_i u_j
\end{aligned}$$

where

$$J_j(a, u_1, \ldots, u_m) = \int\int_S \frac{\partial^2 F}{\partial u^2}\Big(a, \sigma u_j + \tau \sum_{l=j+1}^{m} u_l\Big)\, d\sigma\, d\tau.$$

After dropping the summation and the integral sign, and after dropping all terms that are obviously $O(\epsilon^2)$, we are left with the three terms, in which we note that $i > j$,

$$\epsilon^{-2} \frac{\partial^2 F}{\partial u^2}\Big(V(\epsilon(x-\xi^{(k)})), g_{\sigma,\tau,\epsilon,s,w,k,j}\Big) \phi^{(i)}(x+\delta^{(ki)}) \phi^{(j)}(x+\delta^{(kj)}),$$

$$\frac{\partial^2 F}{\partial u^2}\Big(V(\epsilon(x-\xi^{(k)})), g_{\sigma,\tau,\epsilon,s,w,k,j}\Big) w^{(i)}(x+\delta^{(ki)}) \phi^{(j)}(x+\delta^{(kj)})$$

and

$$\frac{\partial^2 F}{\partial u^2}\Big(V(\epsilon(x-\xi^{(k)})), g_{\sigma,\tau,\epsilon,s,w,k,j}\Big) \phi^{(i)}(x+\delta^{(ki)}) w^{(j)}(x+\delta^{(kj)}),$$

where

$$g_{\sigma,\tau,\epsilon,s,w,k,j} = \sigma(\phi^{(j)} + \epsilon^2 w^{(j)})(x + \delta^{(kj)}) + \tau \sum_{l=j+1}^{m} (\phi^{(l)} + \epsilon^2 w^{(l)})(x + \delta^{(kl)})$$

is uniformly bounded in H^2 for $0 \le \sigma, \tau \le 1$, $\epsilon \to 0^+$, $s \in \mathbf{R}^n$, w in a bounded set. That these tend to 0 in L^2 follows from the same arguments as in the corresponding place in the previous section, using the exponential decay of ϕ_a for the first term, and property (F5) for the second and third, together with $\delta^{(ki)} - \delta^{(kj)} = \delta^{(ji)} = O(1/\epsilon)$.

Now we consider the derivative of the coupling terms of the rescaled problem w.r.t. s. This tends to 0 in norm, as $s \to 0$, $w \to 0$, $\epsilon \to 0^+$, or, equivalently, the partial derivatives w.r.t. the coordinates of $\xi^{(k)}$ and $\delta^{(ki)}$ of the kth equation tend to 0 in L^2. Referring to the corresponding place in the previous section we have to show that the following tend to 0 in L^2:

$$\epsilon^{-1} D_l V(\epsilon(x - \xi^{(k)}))\phi^{(i)}(x + \delta^{(ki)})\phi^{(j)}(x + \delta^{(kj)}),$$

$$\epsilon^{-2} \frac{\partial^2 F}{\partial u^2}\Big(V(\epsilon(x - \xi^{(k)})), h_{\tau,\epsilon,s,w,k,j}\Big) D_l \phi^{(i)}(x + \delta^{(ki)})\phi^{(j)}(x + \delta^{(kj)}),$$

$$\frac{\partial^2 F}{\partial u^2}\Big(V(\epsilon(x - \xi^{(k)})), h_{\tau,\epsilon,s,w,k,j}\Big) D_l w^{(i)}(x + \delta^{(ki)})\phi^{(j)}(x + \delta^{(kj)})$$

and

$$\frac{\partial^2 F}{\partial u^2}\Big(V(\epsilon(x - \xi^{(k)})), h_{\tau,\epsilon,s,w,k,j}\Big) D_l \phi^{(i)}(x + \delta^{(ki)}) w^{(j)}(x + \delta^{(kj)})$$

where $h_{\tau,\epsilon,s,w,k,j}$ is bounded in H^2 for $0 \le \tau \le 1$, $\epsilon \to 0^+$, $s \in \mathbf{R}^n$ and w in a bounded set. For the first we use (V1) (with the fact that b_k is a critical point of V) and (Φ3). For the remaining three we use (F5.1).

Next we consider the verification of condition 5 of the modified implicit function theorem. Let us denote the derivative of the rescaled problem at (s, w) by $\Gamma_\epsilon(s, w)$. We have that

$$\begin{aligned}\Gamma_\epsilon^{(k)}(s,w)(\sigma,v) = -\triangle v^{(k)} &+ \frac{\partial F}{\partial u}\Big(V(\epsilon(x-\xi^{(k)})), \phi^{(k)}\Big) v^{(k)} \\ &+ \lambda_k \frac{\partial F}{\partial u}\Big(V(\epsilon(x-\xi^{(k)})), \sum_{i=1}^{m}\phi^{(i)}(x+\delta^{(ki)})\Big) \sum_{i=1}^{m} v^{(i)}(x+\delta^{(ki)}) \\ &- \lambda_k \sum_{i=1}^{m} \frac{\partial F}{\partial u}\Big(V(\epsilon(x-\xi^{(k)})), \phi^{(i)}(x+\delta^{(ki)})\Big) v^{(i)}(x+\delta^{(ki)}) \\ &+ \Gamma_\epsilon^{(k)}(s,w)(\sigma,0) + O(\epsilon^2). \qquad (5.3)\end{aligned}$$

We suppose that there exist sequences

$$\epsilon_\nu \to 0, \quad \sigma_\nu \in (\mathbf{R}^n)^m, \quad v_\nu \in W_1 \times \ldots \times W_m$$

such that

$$||\sigma_\nu|| + ||v_\nu||_{2,2} = 1$$

whilst

$$\Gamma_{\epsilon_\nu}(0,\eta)(\sigma_\nu, v_\nu) \to 0$$

in $(L^2)^m$, and deduce a contradiction. Without loss of generality we may assume that $\sigma_\nu \to \sigma$ and the limits

$$v_\nu^{(i)}(\cdot + \delta_\nu^{(ji)}) \to y^{(ij)}, \quad 1 \le i,j \le m$$

hold in the weak topology on H^2. Note that $y^{(ii)} \in W_i$.

Proceeding to the distribution limit using (5.3), firstly after replacing x by $x + \delta_\nu^{(jk)}$, and secondly without such replacement, we find

$$- \triangle y^{(kj)} \; + \; \frac{\partial F}{\partial u}\Big(a_j, 0\Big) y^{(kj)} \; + \; \lambda_k \left(\frac{\partial F}{\partial u}\Big(a_j, \phi^{(j)}\Big) - \frac{\partial F}{\partial u}\Big(a_j, 0\Big) \right) \sum_{\substack{i=1 \\ i \ne j}}^{m} y^{(ij)} \; = \; 0 \quad (5.4)$$

for $k \neq j$ whereas

$$- \triangle y^{(kk)} + \frac{\partial F}{\partial u}\Big(a_k, \phi^{(k)}\Big) y^{(kk)} + \lambda_k \left(\frac{\partial F}{\partial u}\Big(a_k, \phi^{(k)}\Big) - \frac{\partial F}{\partial u}\Big(a_k, 0\Big) \right) \sum_{\substack{i=1 \\ i \ne k}}^{m} y^{(ik)}$$
$$- \frac{\partial F}{\partial a}\Big(a_k, \phi^{(k)}\Big) H(b_k) x \cdot \sigma^{(k)} = 0. \quad (5.5)$$

Let $z_j = \sum_{k=1, k\neq j}^{m} y^{(kj)}$. Summing (5.4) over k we deduce

$$-\triangle z_j + \frac{\partial F}{\partial u}\Big(a_j, 0\Big) z_j + \mu_j \left(\frac{\partial F}{\partial u}\Big(a_j, \phi^{(j)}\Big) - \frac{\partial F}{\partial u}\Big(a_j, 0\Big) \right) z_j = 0$$

where $\mu_j = \sum_{k=1, k\neq j}^{m} \lambda_k$. Owing to condition (P1) and Lemma 3.5 there is a discrete set $D_j \subset \mathbf{R}$ such that the operator

$$-\triangle + \frac{\partial F}{\partial u}\Big(a_j, 0\Big) + \mu_j \left(\frac{\partial F}{\partial u}\Big(a_j, \phi^{(j)}\Big) - \frac{\partial F}{\partial u}\Big(a_j, 0\Big) \right)$$

is invertible from H^2 to L^2 for $\mu_j \notin D_j$. The hyperplanes $\sum_{k=1, k\neq j}^{m} \lambda_k \in D_j$, $j = 1, \dots, m$, meet the hyperplane $\sum_{k=1}^{m} \lambda_k = 1$ in a nowhere dense set. We conclude that there is a dense subset C of the hyperplane $\sum_{k=1}^{m} \lambda_k = 1$ such that for $(\lambda_1, \dots, \lambda_m) \in C$ the equations (5.4) and (5.5) imply that $z_j = 0$ for all j, and hence $y^{(ij)} = 0$, for all i and j, and $\sigma = 0$.

For such a choice of λ_k the assumption that $\Gamma_{\epsilon_\nu}(0,\eta)(\sigma_\nu, v_\nu) \to 0$ in L^2, together with the formula (5.3), repeated application of Lemma 3.5 combined with judicious shifting, leads to the conclusion

$$-\triangle v_\nu^{(k)} + \frac{\partial F}{\partial u}\Big(V(\epsilon(x-\xi_\nu^{(k)})),0\Big)v_\nu^{(k)} \to 0$$

in L^2. Now Wang's Lemma implies the contradiction $||v_\nu||_{2,2} \to 0$.

The verification of condition 6 of the MIFT is carried out as in section 4.

6 Standard growth conditions

The conditions imposed on the function F in sections 3 and 4 follow from growth conditions (we are not claiming that these conditions are necessary, though this is plausible). These take the form

$$|F(a,u)|,\ \left|\frac{\partial F}{\partial a}\Big(a,u\Big)\right|,\ \left|\frac{\partial^2 F}{\partial a^2}\Big(a,u\Big)\right| \le C(|u|+|u|^{\alpha_1}),$$

$$\left|\frac{\partial F}{\partial u}\Big(a,u\Big)\right|,\ \left|\frac{\partial^2 F}{\partial u \partial a}\Big(a,u\Big)\right| \le C(1+|u|^{\alpha_2}),$$

$$\left|\frac{\partial^2 F}{\partial u^2}\Big(a,u\Big)\right| \le C(1+|u|^{\alpha_3}),$$

where the constant C can be chosen uniformly for a in a bounded interval and the exponents are non-negative and satisfy $\alpha_k \ge 2-k$.

These are called *standard growth conditions* if, in addition, $1 \le n \le 7$, with no upper limit placed on $\alpha_1, \alpha_2, \alpha_3$ if $n \le 4$, whereas for $n = 5, 6, 7$ we assume

$$\alpha_1 \le \frac{n}{n-4}, \quad \alpha_2 \le \frac{4}{n-4}, \quad \alpha_3 < \frac{8-n}{n-4}.$$

The inequality for α_3 is strict in order to obtain at the appropriate point a compact embedding.

Theorem 6.1 *Assume that F has continuous second-order partial derivatives and satisfies the standard growth conditions. Then F satisfies conditions (F1–6) of section 3.*

Proof (F2), (F3) and (F5) were dealt with in [12]. (F6) is obvious. Consider finally (F4). Let m_ν be a bounded sequence in L^∞, let $u \in H^2$ and let h_ν be a bounded sequence in L^∞ converging pointwise to 0. Then we have the pointwise bound

$$\left|\frac{\partial F}{\partial a}\Big(m_\nu, u\Big)h_\nu\right| \le C(|u|.|h_\nu| + |u|^{\alpha_1}|h_\nu|) \le \text{const.}(|u|+|u|^{\alpha_1})$$

Hence $\frac{\partial F}{\partial a}(m_\nu, u)h_\nu$ tends to 0 pointwise and is uniformly bounded by a function in L^2. Hence it converges to 0 in L^2. The other parts of (F4) are treated similarly.

Theorem 6.2 *Suppose that F satisfies the standard conditions with the following additional restrictions:*

(1) $\dfrac{\partial^3 F}{\partial^2 u\, \partial a}$ *exists and is continuous;*

(2) $n \leq 5$, *and if* $n = 5$ *then* $\alpha_2 \leq 2$ *and* $\alpha_3 < 1$.

Then F satisfies (F1.1), (F2.1), (F3.1) and (F5.1).

Theorems 6.1 and 6.2 are applications of the following two lemmas (see [12]). Let V_i denote a family of function spaces drawn from the collection $W^{k,p}(\mathbf{R}^n)$. Let us suppose that V_i is embedded into L^r for $2 \leq r \leq t_i$.

Lemma 6.3 *Let $u_i \in V_i$, $i = 1, \ldots, m$. Then $u_1^{\beta_1} \ldots u_m^{\beta_m} \in L^r$ if*

$$\frac{\beta_1}{t_1} + \cdots + \frac{\beta_m}{t_m} \leq \frac{1}{r} \leq \frac{\beta_1 + \cdots + \beta_m}{2}$$

and

$$||u_1^{\beta_1} \ldots u_m^{\beta_m}||_r \leq C||u_1||_{V_1}^{\beta_1} \ldots ||u_m||_{V_m}^{\beta_m}$$

where C is independent of u_k.

Lemma 6.4 *Suppose that $V_1 = W^{k,p}$ for some $k \geq 1$. Let $w \in V_2$. Then the linear map $v \to vw$ from V_1 to L^r is compact for*

$$\frac{1}{t_1} + \frac{1}{t_2} < \frac{1}{r} \leq 1.$$

Using Lemma 6.3 one can easily show that under the conditions of Theorem 6.2 properties (F2.1) and (F3.1) hold. Lemma 6.4 yields (F5.1).

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