Non-degeneracy of perturbed solutions of semilinear partial differential equations

Robert Magnus, Olivier Moschetta*

Abstract

The equation $-\Delta u + F(V(\varepsilon x), u) = 0$ is considered in $\mathbb{R}^n$. For small $\varepsilon > 0$ it is shown to possess, under appropriate conditions, a non-degenerate solution u_ε in $H^2(\mathbb{R}^n)$. It is shown that the linearised operator T_ε at the solution satisfies $\|T_\varepsilon^{-1}\| = O(\varepsilon^{-2})$ as $\varepsilon \to 0$.

1 Introduction

In this paper we consider the question of non-degeneracy of certain solutions of a partial differential equation, where by non-degenerate we mean that the linearised problem at the solution, in appropriate function spaces, defines an invertible linear operator. Throughout this article, by an invertible operator, with specified Banach spaces as domain and codomain, we shall mean a linear surjective homeomorphism.

The importance of non-degenerate solutions lies in their stable behaviour under perturbations. Indeed the implicit function theorem implies the persistence of a non-degenerate solution under perturbations of the problem that are C^1-small and map between the same function spaces. Moreover for equations of the type we shall consider, non-degenerate solutions give rise to multibump solutions, see for example [1], [3].

Let us consider first the problem

$$-\Delta u + F(u) = 0 \tag{1.1}$$

in $\mathbb{R}^n$. Under conditions on F to be specified the non-linear operator $\Gamma(u) = -\Delta u + F(u)$ is well-defined from $W^{2,2}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ and will have a well-defined Frechet derivative, the linear operator $v \mapsto D\Gamma(u)v = -\Delta v + \frac{dF}{du}(u)v$. Throughout this paper we shall write H^k for $W^{k,2}(\mathbb{R}^n)$ and L^p for $L^p(\mathbb{R}^n)$.

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Let us assume that we have a solution ϕ of (1.1). It is highly implausible for ϕ to be a non-degenerate solution as the partial derivatives $D_k\phi$ will be in the kernel of $D\Gamma(\phi)$ if they lie in H^2. Similarly, since the problem (1.1) commutes with rotations, we expect the functions $\nabla\phi(x)\cdot Tx$ to be in the kernel whenever T is a skew-symmetric matrix. However these functions will be 0 if ϕ is spherically symmetric.

We shall say that a spherically symmetric solution ϕ is quasi-non-degenerate if the partial derivatives $D_j\phi(x)$ belong to H^2, they are linearly independent, span the kernel of $D\Gamma(\phi)$, whilst the range of $D\Gamma(\phi)$ is the orthogonal complement in L^2 to its kernel.

Quasi-non-degenerate solutions are easy to construct in one dimension. We consider $-u'' + F(u) = 0$ where F is a smooth function such that $F(0) = 0$, $F'(0) > 0$ and $\Phi(u) = -\int F(u)\,du$ satisfies $\sup_{u>0}\Phi(u) > \Phi(0)$. Then the solution $\phi(x)$ is quasi-non-degenerate where $x \mapsto (\phi(x), \phi'(x))$ is the phase plane trajectory in the region $u > 0$ which tends to the saddle point $(0,0)$ as $x \to \pm\infty$.

In higher dimensions a range of quasi-non-degenerate solutions is known for the equation

$$-\Delta u + u - u^p = 0 \tag{1.2}$$

More precisely it is known that the ground state solution, defined to be the solution with minimum energy $\int(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2 - \frac{1}{p+1}u^{p+1})\,dx$, exists and is quasi-non-degenerate for all $p > 1$ if $n = 1, 2$ and for $1 < p < (n+2)/(n-2)$ if $n > 2$. See the papers [2], [5], [7].

The possibility arises of obtaining non-degenerate solutions by perturbing (1.1), when a quasi-non-degenerate solution is known, to a problem explicitly containing x. Various perturbation schemes have been studied, for example that of [3, sections 4, 5], which generates non-degenerate solutions to a more general equation of the type $-\Delta u + F(x, u, \nabla u) = 0$. Another paper [4] considered perturbations of a different nature. Suppose we have a one-parameter continuum of problems

$$-\Delta u + F(a, u) = 0 \tag{1.3}$$

where a belongs to real interval I, and suppose for each value $a \in I$ we have a quasi-non-degenerate solution $\phi_a(x)$. Now we perturb (1.3) to

$$-\Delta u + F(V(\varepsilon x), u) = 0 \tag{1.4}$$

where $V(x)$ is function with range in I and $\varepsilon > 0$. The difficulty of this scheme arises from the weak nature of the convergence to a problem of the form (1.3) as $\varepsilon \to 0$.

The scheme just described acquires an added interest from the observation that if $\psi(x)$ is a solution of (1.2) then $\phi_a(x) = a^\mu\psi(a^\nu x)$ satisfies

$$-\Delta u + a^s u - a^t u^p = 0$$

if $\mu = \frac{s-t+1}{p-1}$ and $\nu = \frac{s}{2}$.

In the previous paper [4, section 3] it was shown how to obtain solutions of (1.4) using rescaling of the unknown function u. The nature of the rescaling obscured the question of non-degeneracy of the perturbed solutions. The main object of this paper is provide a clear proof of non-degeneracy for solutions of (1.4) together with an estimate of the blow-up of the inverse of the derivative as $\varepsilon \to 0$.

2 Principal assumptions

Properties of F.

We assume that F is a C^2 map satisfying the following growth conditions:

$$\begin{aligned}
|F(a,u)|\,,\ \left|\frac{\partial F}{\partial a}(a,u)\right|\,,\ \left|\frac{\partial^2 F}{\partial a^2}(a,u)\right| &\leq C\left(|u| + |u|^{\alpha_1}\right)\\
\left|\frac{\partial F}{\partial u}(a,u)\right|\,,\ \left|\frac{\partial^2 F}{\partial u \partial a}(a,u)\right| &\leq C\left(1 + |u|^{\alpha_2}\right)\\
\left|\frac{\partial^2 F}{\partial u^2}(a,u)\right| &\leq C\left(1 + |u|^{\alpha_3}\right)
\end{aligned}$$

where C is chosen uniformly for a in a bounded interval and the exponents α_i are non-negative (except $\alpha_1 \geq 1$). No upper limits are placed on α_i if $n \leq 4$ whereas for $n \geq 5$ we assume that

$$\alpha_1 \leq \frac{n}{n-4}, \quad \alpha_2 \leq \frac{4}{n-4}, \quad \alpha_3 < \frac{8-n}{n-4}$$

Under these growth conditions F, $\dfrac{\partial F}{\partial a}$, $\dfrac{\partial^2 F}{\partial a^2}$, $\dfrac{\partial F}{\partial u}$, $\dfrac{\partial^2 F}{\partial u \partial a}$ and $\dfrac{\partial^2 F}{\partial u^2}$ define Nemitskii operators

$$\begin{aligned}
\boldsymbol{F},\ \boldsymbol{F}_a,\ \boldsymbol{F}_{aa} &: L^\infty \times H^2 \to L^2\\
\boldsymbol{F}_u,\ \boldsymbol{F}_{ua} &: L^\infty \times H^2 \to \mathcal{L}(H^2, L^2)\\
\boldsymbol{F}_{uu} &: L^\infty \times H^2 \to \mathcal{L}_2(H^2 \times H^2, L^2)
\end{aligned}$$

by means of

$$\boldsymbol{F}(m,u) = F(m,u)\,,\ \boldsymbol{F}_u(m,u)v = \frac{\partial F}{\partial u}(m,u)v$$

and so on. In these formulas we use $\mathcal{L}_k$ to denote the appropriate space of symmetric k-linear mappings.

Under these conditions, the Nemitskii operators induced by F and its derivatives have the following boundedness property (see [4]):

Lemma 2.1 *The maps $\boldsymbol{F}$, $\boldsymbol{F}_a$, $\boldsymbol{F}_{aa}$, $\boldsymbol{F}_u$, $\boldsymbol{F}_{ua}$ and $\boldsymbol{F}_{uu}$ map bounded subsets of $L^\infty \times H^2$ to bounded subsets of the appropriate function or operator space.*

The following convergence properties were proved in [4] and will be used repeatedly later on.

Lemma 2.2 *Let $m_\nu \in L^\infty$ be a bounded sequence that tends pointwise to $m \in L^\infty$. Let u_ν in H^2 converge to $u \in H^2$ and let $v, w \in H^2$. Then*

$$\boldsymbol{F}(m_\nu, u_\nu) \to \boldsymbol{F}(m, u), \ \boldsymbol{F}_a(m_\nu, u_\nu) \to \boldsymbol{F}_a(m, u), \ \boldsymbol{F}_u(m_\nu, u_\nu)v \to \boldsymbol{F}_u(m, u)v$$

$$\boldsymbol{F}_{uu}(m_\nu, u_\nu)(v, w) \to \boldsymbol{F}_{uu}(m, u)(v, w)$$

Lemma 2.3 *Let $m_\nu \in L^\infty$ be a bounded sequence that tends pointwise to $m \in L^\infty$ and let $u_\nu \in H^2$ converge weakly to $u \in H^2$. Then, for any bounded sequence $v_\nu \in H^2$,*

$$\boldsymbol{F}_u(m_\nu, u_\nu)v_\nu - \boldsymbol{F}_u(m, u)v_\nu \longrightarrow 0$$

in the weak topology on the dual of H^2.

Lemma 2.4 *Let m_ν be a bounded family in L^∞, and u_ν, v_ν and w_ν be bounded sequences in H^2 such that either*

1. *$u_\nu - v_\nu$ is convergent in H^2 and w_ν converges weakly to 0, or*
2. *$u_\nu - v_\nu$ converges weakly to 0 in H^2 and w_ν is convergent.*

Then

$$\big(\boldsymbol{F}_u(m_\nu, u_\nu) - \boldsymbol{F}_u(m_\nu, v_\nu)\big)w_\nu \to 0$$

in L^2. Furthermore, $\boldsymbol{F}_u(m, u) - \boldsymbol{F}_u(m, v)$ is a compact operator for each $m \in L^\infty$, and u, $v \in H^2$.

Properties of ϕ_a.

The function $\phi_a(x)$ is a solution to $-\Delta u + F(a, u) = 0$ in H^2 and has the following properties:

1. $\phi_a(x) = \Phi_a(r)$ is spherically symmetric.
2. $\displaystyle\int \frac{\partial F}{\partial a}(a, \Phi_a(r))\, \Phi_a'(r) r \, dx \neq 0.$
3. ϕ_a and its first derivatives have exponential decay.
4. ϕ_a is a quasi-non-degenerate solution, that is, the operator

$$-\Delta + \frac{\partial F}{\partial u}(a, \phi_a(x)) : H^2 \to L^2$$

has as its kernel the space spanned by the n partial derivatives $D_j \phi_a(x)$, which are assumed to be independent, and its range is the space in L^2 orthogonal to its kernel. This implicitly says that the partial derivatives belong to H^2.

These properties hold in the model case of the non-linear Schrödinger equation 1.2 described in the introduction, see [2], [5] and [7].

Properties of V.
The function V is C^2 with its range in the interval I. It and its first partial derivatives are bounded, while its second partial derivatives have polynomial growth.

Positivity assumption.
There exists $\delta > 0$ such that

$$\frac{\partial F}{\partial u}(a, 0) > \delta$$

for all a in the range of V.

For later reference we shall need a version of Wang's Lemma (see [6, 3]):

Lemma 2.5 *Let f_ν be a family of measurable functions such that*

$$0 < \delta < f_\nu(x) < K$$

for all ν and constants δ and K. Let μ_ν be a sequence of non-negative numbers and let v_ν be a sequence in H^2 such that

$$-\Delta v_\nu + (f_\nu(x) + \mu_\nu)v_\nu \to 0$$

in L^2. Then $v_\nu \to 0$ in H^2.

Under these conditions the following theorem, proved in [4], holds.

Theorem 2.6 *Let b be a non-degenerate critical point of V and let $a = V(b)$. Then, for sufficiently small $\varepsilon > 0$, the equation $-\Delta u + F(V(\varepsilon x), u) = 0$ has a solution of the form*

$$u_\varepsilon(x) = \phi_a\left(x - \frac{b}{\varepsilon} + s_\varepsilon\right) + \varepsilon^2 w_\varepsilon\left(x - \frac{b}{\varepsilon} + s_\varepsilon\right)$$

where $s_\varepsilon \in \mathbb{R}^n$, $w_\varepsilon \in H^2$ and w_ε is orthogonal in L^2 to the partial derivatives $D_j\phi_a$. Both s_ε and w_ε depend continuously on ε. As ε tends to 0, s_ε tends to 0 and w_ε tends to a computable function $\eta \in H^2$, which is the unique solution $v = \eta(x)$ orthogonal to the partial derivatives $D_j\phi_a$ of the problem

$$-\Delta v + \frac{\partial F}{\partial u}(a, \phi_a(x))v = -\frac{1}{2}\frac{\partial F}{\partial a}(a, \phi_a(x))(H(b)x \cdot x) \tag{2.1}$$

where $H(b)$ is the Hessian matrix of V at the point b.

3 Non-degeneracy of the solutions

The main conclusion of this paper is the following result.

Theorem 3.1 *For sufficiently small $\varepsilon > 0$ the solutions u_ε obtained under the conditions of theorem 2.6 are non-degenerate, that is, the operator*

$$T_\varepsilon := -\Delta + \frac{\partial F}{\partial u}(V(\varepsilon x), u_\varepsilon)$$

from H^2 to L^2 is invertible. Moreover, we have the following bound on its inverse:

$$\|T_\varepsilon^{-1}\| = O\left(\frac{1}{\varepsilon^2}\right)$$

as ε tends to 0.

Proof. Checking invertibility of an operator and getting estimates on the norm of its inverse is always made easier by knowing that this operator is Fredholm, so we first remark that our positivity assumption implies that T_ε is a Fredholm operator of index 0. Indeed, let

$$A_\varepsilon = -\Delta + \frac{\partial F}{\partial u}(V(\varepsilon x), 0)$$

from H^2 to L^2. By the positivity assumption A_ε is a self-adjoint operator with domain H^2 satisfying $A_\varepsilon > -\Delta + \delta$, and hence an invertible operator from H^2 to L^2. Now T_ε is a compact perturbation of A_ε, since $T_\varepsilon - A_\varepsilon$ is given by multiplication by

$$f(x) := \frac{\partial F}{\partial u}(V(\varepsilon x), u_\varepsilon) - \frac{\partial F}{\partial u}(V(\varepsilon x), 0)$$

and therefore defines a compact operator from H^2 to L^2 by lemma 2.4.

Now we have a useful criterion that gives invertibility and our sought for estimate on the inverse.

Lemma 3.2 *Let $(\Gamma_\varepsilon)_{0 \le \varepsilon \le \varepsilon_0}$ be a family of Fredholm operators of index 0 between two Banach spaces E and F. Suppose that there* ***do not exist*** *sequences $\varepsilon_\nu \to 0$, $x_\nu \in E$, such that $\|x_\nu\| = 1$ and $\Gamma_{\varepsilon_\nu} x_\nu \to 0$. There then exists $\varepsilon_1 > 0$, such that for all $0 < \varepsilon < \varepsilon_1$, the operator Γ_ε is invertible and there exists a constant K independent of ε such that $\|\Gamma_\varepsilon^{-1}\| \le K$.*

Proof of lemma 3.2. Under our assumption there is no sequence $x_\nu \in \mathrm{Ker}(\Gamma_{\varepsilon_\nu})$ with $\|x_\nu\| = 1$ and $\varepsilon_\nu \to 0$; so Γ_ε is injective for sufficiently small ε, and therefore invertible because of the Fredholm alternative.

Assume next (seeking a contradiction) that we can find a sequence $\varepsilon_\nu \to 0$ such that $\|\Gamma_{\varepsilon_\nu}^{-1}\| \to +\infty$. Then there exists a sequence $y_\nu \in F$, such that $\|y_\nu\| = 1$ and $\|\Gamma_{\varepsilon_\nu}^{-1} y_\nu\| \to +\infty$. Letting

$$x_\nu = \frac{\Gamma_{\varepsilon_\nu}^{-1} y_\nu}{\|\Gamma_{\varepsilon_\nu}^{-1} y_\nu\|} \in E$$

we see that $\|x_\nu\| = 1$ and

$$\Gamma_{\varepsilon_\nu} x_\nu = \frac{y_\nu}{\|\Gamma_{\varepsilon_\nu}^{-1} y_\nu\|}$$

which tends to 0, contradicting the assumption of the lemma.

After these preliminaries, we can continue with the proof of theorem 3.1. Let

$$S_\varepsilon = -\Delta + \frac{\partial F}{\partial u}(V(\varepsilon(x - s_\varepsilon) + b), \phi_a + \varepsilon^2 w_\varepsilon) : H^2 \to L^2.$$

Then $S_\varepsilon = R_\varepsilon^{-1} T_\varepsilon R_\varepsilon$ where R_ε is translation

$$(R_\varepsilon v)(x) = v\left(x + s_\varepsilon - \frac{b}{\varepsilon}\right),$$

which, considered either from H^2 to itself or L^2 to itself, is an isometry. It suffices therefore to show that S_ε is invertible and that $\|S_\varepsilon^{-1}\| = O(\varepsilon^{-2})$.

According to lemma 3.2 we seek a contradiction from the assumption that there exist sequences $\varepsilon \to 0$ and $\widetilde{u_\varepsilon} \in H^2$, with $\|\widetilde{u_\varepsilon}\|_{H^2} = 1$, such that

$$\varepsilon^{-2}\left(-\Delta \widetilde{u_\varepsilon} + \frac{\partial F}{\partial u}(V(\varepsilon(x - s_\varepsilon) + b), \phi_a + \varepsilon^2 w_\varepsilon)\widetilde{u_\varepsilon}\right) \to 0 \tag{3.1}$$

in L^2 as ε tends to 0. (For the sake of readability we drop the index ν from now on with the understanding that by $\varepsilon \to 0$ we mean a sequence $\varepsilon_\nu \to 0$.)

Let $\widetilde{u_\varepsilon} = \sigma_\varepsilon \cdot \nabla\phi_a + \gamma_\varepsilon \widetilde{v_\varepsilon}$ where $\sigma_\varepsilon \in \mathbb{R}^n$, $\widetilde{v_\varepsilon} \in W$, $\|\widetilde{v_\varepsilon}\|_{H^2} = 1$ and $\gamma_\varepsilon \geq 0$. Then γ_ε and σ_ε are bounded sequences and we may assume (going to a subsequence) that $\gamma_\varepsilon \to \gamma_0$, $\sigma_\varepsilon \to \sigma_0$ and $\widetilde{v_\varepsilon} \to v_0$ weakly in H^2.

In view of the equation $(-\Delta + \frac{\partial F}{\partial u}(a, \phi_a))\nabla\phi_a = 0$ (3.1) implies

$$\varepsilon^{-2}\gamma_\varepsilon S_\varepsilon \widetilde{v_\varepsilon} + \varepsilon^{-2}\left[\frac{\partial F}{\partial u}(V(\varepsilon(x - s_\varepsilon) + b), \phi_a + \varepsilon^2 w_\varepsilon) - \frac{\partial F}{\partial u}(a, \phi_a)\right](\nabla\phi_a \cdot \sigma_\varepsilon) \to 0 \tag{3.2}$$

in L^2. We claim that the second term in (3.2) tends in L^2 to

$$\frac{\partial^2 F}{\partial u^2}(a, \phi_a)\eta\, \nabla\phi_a \cdot \sigma_0 + \frac{1}{2}\frac{\partial^2 F}{\partial u \partial a}(a, \phi_a)(H(b)x \cdot x)\nabla\phi_a \cdot \sigma_0$$

where $H(b)$ is the Hessian matrix of V and η is the solution to (2.1) (see theorem 2.6). To see this we expand it into

$$\begin{aligned}\varepsilon^{-2}\left(\frac{\partial F}{\partial u}(V(\varepsilon(x-s_\varepsilon)+b),\phi_a+\varepsilon^2 w_\varepsilon)-\frac{\partial F}{\partial u}(V(\varepsilon(x-s_\varepsilon)+b),\phi_a)\right)(\nabla\phi_a\cdot\sigma_\varepsilon)\\ +\varepsilon^{-2}\left(\frac{\partial F}{\partial u}(V(\varepsilon(x-s_\varepsilon)+b),\phi_a)-\frac{\partial F}{\partial u}(a,\phi_a)\right)(\nabla\phi_a\cdot\sigma_\varepsilon)\end{aligned}\tag{3.3}$$

For the first term of (3.3), we obtain

$$\begin{aligned}\varepsilon^{-2}\left(\frac{\partial F}{\partial u}(V(\varepsilon(x-s_\varepsilon)+b),\phi_a+\varepsilon^2 w_\varepsilon)-\frac{\partial F}{\partial u}(V(\varepsilon(x-s_\varepsilon)+b),\phi_a)\right)(\nabla\phi_a\cdot\sigma_\varepsilon)\\ =\int_0^1\frac{\partial^2 F}{\partial u^2}(V(\varepsilon(x-s_\varepsilon)+b),\phi_a+\tau\varepsilon^2 w_\varepsilon)w_\varepsilon\nabla\phi_a\cdot\sigma_\varepsilon\,d\tau\end{aligned}\tag{3.4}$$

For fixed $\tau\in[0,1]$

$$\frac{\partial^2 F}{\partial u^2}(V(\varepsilon(x-s_\varepsilon)+b),\phi_a+\tau\varepsilon^2 w_\varepsilon)\eta\nabla\phi_a\cdot\sigma_0\longrightarrow\frac{\partial^2 F}{\partial u^2}(a,\phi_a)\eta\nabla\phi_a\cdot\sigma_0$$

according to Lemma 2.2. Moreover,

$$\begin{aligned}\frac{\partial^2 F}{\partial u^2}(V(\varepsilon(x-s_\varepsilon)+b),\phi_a+\tau\varepsilon^2 w_\varepsilon)w_\varepsilon(\nabla\phi_a\cdot\sigma_\varepsilon)\\ -\frac{\partial^2 F}{\partial u^2}(V(\varepsilon(x-s_\varepsilon)+b),\phi_a+\tau\varepsilon^2 w_\varepsilon)\eta(\nabla\phi_a\cdot\sigma_0)\end{aligned}$$

tends to 0 owing to Lemma 2.1, the boundedness of V, and since $w_\varepsilon\to\eta$ in H^2 and $\nabla\phi_a\cdot\sigma_\varepsilon\to\nabla\phi_a\cdot\sigma_0$ in H^2. Therefore, the integrand in (3.4) tends to

$$\frac{\partial^2 F}{\partial u^2}(a,\phi_a)\eta\nabla\phi_a\cdot\sigma_0$$

in L^2 at fixed τ. Also, the L^2-norm of the integrand stays bounded independently of τ and ε (again by Lemma 2.1), so the dominated convergence theorem for L^2-valued integrals shows that the integral tends in L^2 to

$$\frac{\partial^2 F}{\partial u^2}(a,\phi_a)\eta\nabla\phi_a\cdot\sigma_0$$

For the second term of (3.3) we obtain

$$\begin{aligned}&\varepsilon^{-2}\left(\frac{\partial F}{\partial u}(V(\varepsilon(x-s_\varepsilon)+b),\phi_a)-\frac{\partial F}{\partial u}(a,\phi_a)\right)(\nabla\phi_a\cdot\sigma_\varepsilon)\\ &\quad=\varepsilon^{-1}\int_0^1\frac{\partial^2 F}{\partial u\partial a}(V(\tau\varepsilon(x-s_\varepsilon)+b,\phi_a)\big(\nabla\phi_a\cdot\sigma_\varepsilon\big)\nabla V(\tau\varepsilon(x-s_\varepsilon)+b)\cdot(x-s_\varepsilon)\,d\tau\end{aligned}$$

Since, by assumption, $\nabla V(b)=0$ this is equal to

$$\int_0^1\int_0^1 \frac{\partial^2 F}{\partial u\partial a}(V(\tau\varepsilon(x-s_\varepsilon)+b,\phi_a)\big(\nabla\phi_a\cdot\sigma_\varepsilon\big)H(\rho\tau\varepsilon(x-s_\varepsilon)+b)(x-s_\varepsilon)\cdot(x-s_\varepsilon)\tau\,d\rho\,d\tau$$

Now, $H(x)$ has polynomial growth, $\frac{\partial^2 F}{\partial u\partial a}(V(\tau\varepsilon(x-s_\varepsilon)+b,\phi_a)$ is bounded uniformly with respect to τ and ε as x goes to infinity because of our growth conditions, and $(\nabla\phi_a\cdot\sigma_\varepsilon)$ has uniform exponential decay as x goes to infinity (since σ_ε is a bounded sequence). Hence, for fixed τ, the integrand converges to

$$\frac{\partial^2 F}{\partial u\partial a}(a,\phi_a)\big(\nabla\phi_a\cdot\sigma_0\big)H(b)(x\cdot x)\tau$$

in L^2, as ε goes to 0, and is also bounded uniformly with respect to τ and ε by a fixed function in L^2. We can therefore apply the dominated convergence theorem for L^2-valued integrals, and the double integral tends to

$$\frac{1}{2}\frac{\partial^2 F}{\partial u\partial a}(a,\phi_a)\big(\nabla\phi_a\cdot\sigma_0\big)H(b)(x\cdot x).$$

This proves our claim about the second term of (3.2).

Next we consider the first term of (3.2). We claim that $\varepsilon^{-2}\gamma_\varepsilon$ is bounded. Suppose, for the sake of contradiction, that $\varepsilon^{-2}\gamma_\varepsilon$ is unbounded. Going to a subsequence we may assume that $\varepsilon^2\gamma_\varepsilon^{-1}\to 0$ and multiplying (3.2) by it we obtain $S_\varepsilon\widetilde{v}_\varepsilon\to 0$, that is,

$$-\Delta\widetilde{v}_\varepsilon+\frac{\partial F}{\partial u}(V(\varepsilon(x-s_\varepsilon)+b),\phi_a+\varepsilon^2 w_\varepsilon)\widetilde{v}_\varepsilon\to 0 \tag{3.5}$$

in L^2. Now we have

$$\frac{\partial F}{\partial u}(a,\phi_a)\widetilde{v}_\varepsilon\to\frac{\partial F}{\partial u}(a,\phi_a)v_0$$

weakly in L^2 since multiplication by $\frac{\partial F}{\partial u}(a,\phi_a)$ is a norm-continuous linear operator from H^2 to L^2. Moreover

$$\left(\frac{\partial F}{\partial u}(V(\varepsilon(x-s_\varepsilon)+b),\phi_a+\varepsilon^2 w_\varepsilon)-\frac{\partial F}{\partial u}(a,\phi_a)\right)\widetilde{v}_\varepsilon\to 0$$

in the weak topology of the dual of H^2 by lemma 2.3. Finally $\Delta\widetilde{v}_\varepsilon\to\Delta v_0$ in the sense of distributions. We conclude from (3.5) that

$$-\Delta v_0+\frac{\partial F}{\partial u}(a,\phi_a)v_0=0.$$

But $v_0\in W$ so that $v_0=0$. We deduce by lemma 2.4 that

$$\left(\frac{\partial F}{\partial u}(V(\varepsilon(x-s_\varepsilon)+b),\phi_a+\varepsilon^2 w_\varepsilon)-\frac{\partial F}{\partial u}(V(\varepsilon(x-s_\varepsilon)+b),\phi_a)\right)\widetilde{v}_\varepsilon\to 0$$

in L^2, so that now (3.5) yields

$$\left(-\Delta + \frac{\partial F}{\partial u}(V(\varepsilon(x - s_\varepsilon) + b), 0)\right) \widetilde{v}_\varepsilon \to 0$$

in L^2. Now the positivity assumption and Wang's lemma give $\widetilde{v}_\varepsilon \to 0$ in H^2 thus contradicting the assumption that $\|\widetilde{v}_\varepsilon\|_{H^2} = 1$. This contradiction implies that $\varepsilon^{-2}\gamma_\varepsilon$ is bounded as claimed.

We may therefore assume, going once more to a subsequence, that $\varepsilon^{-2}\gamma_\varepsilon \to c \geq 0$. Armed with this knowledge we return to (3.2) reminding ourselves at this point that we do not have $v_0 = 0$. We do, however, still have that

$$S_\varepsilon \widetilde{v}_\varepsilon \to -\Delta v_0 + \frac{\partial F}{\partial u}(a, \phi_a) v_0$$

in the sense of distributions. Passing to the limit in 3.2 we find that

$$\begin{aligned}\left(-\Delta + \frac{\partial F}{\partial u}(a, \phi_a)\right) c v_0 + \frac{\partial^2 F}{\partial u^2}(a, \phi_a)\eta(\nabla\phi_a \cdot \sigma_0) \\ + \frac{1}{2}\frac{\partial^2 F}{\partial u \partial a}(a, \phi_a)(H(b)x \cdot x)\nabla\phi_a \cdot \sigma_0 = 0 \quad (3.6)\end{aligned}$$

Now recall that η satisfies the equation

$$-\Delta\eta + \frac{\partial F}{\partial u}(a, \phi_a)\eta = -\frac{1}{2}\frac{\partial F}{\partial a}(a, \phi_a)(H(b)x \cdot x)$$

Differentiating this with respect to x gives the vector-valued equation

$$\begin{aligned}-\Delta(\nabla\eta) + \frac{\partial F}{\partial u}(a, \phi_a)\nabla\eta + \frac{\partial^2 F}{\partial u^2}(a, \phi_a)\eta\nabla\phi_a \\ = -\frac{1}{2}\frac{\partial^2 F}{\partial a \partial u}(a, \phi_a)(H(b)x \cdot x)\nabla\phi_a - \frac{\partial F}{\partial a}(a, \phi_a)H(b)x\end{aligned}$$

Taking the inner product with σ_0 and using (3.6) gives

$$\left(-\Delta + \frac{\partial F}{\partial u}(a, \phi_a)\right)(c v_0 + \nabla\eta \cdot \sigma_0) = -\frac{\partial F}{\partial a}(a, \phi_a)H(b)x \cdot \sigma_0 \qquad (3.7)$$

Hence $\frac{\partial F}{\partial a}(a, \phi_a)H(b)x \cdot \sigma_0$ is in the range of $-\Delta + \frac{\partial F}{\partial u}(a, \phi_a)$, which is by assumption W. It follows that σ_0 satisfies the linear equation system

$$\int \left(\frac{\partial F}{\partial a}(a, \phi_a)H(b)x \cdot \sigma_0\right) D_j\phi_a(x)\, dx = 0, \quad j = 1, \ldots, n.$$

We claim that this implies $\sigma_0 = 0$. Write $\sigma_0 = (\sigma_1, \ldots, \sigma_n)$. The inner product $H(b)x \cdot \sigma_0$ is given by

$$H(b)x \cdot \sigma_0 = \sum_{i,k} \sigma_i \left(x_k H_{k,i} \right)$$

where the $H_{k,i}$ are the coefficients of the matrix $H(b)$. Moreover ϕ_a is spherically symmetric, $\phi_a(x) = \Phi_a(r)$, so our system can be written as

$$\int \left[\sum_{i,k} \sigma_i \left(x_k H_{k,i} \right) \right] \frac{\partial F}{\partial a}(a, \Phi_a(r)) \frac{\Phi_a'(r) x_j}{r} \, dx = 0, \quad j = 1, \ldots, n.$$

Spherical symmetry causes terms involving mixed products $x_j x_k$, $j \neq k$, to vanish, leading to the simpler equation

$$\left(\sum_i \sigma_i H_{j,i} \right) \int \frac{\partial F}{\partial a}(a, \Phi_a(r)) \Phi_a'(r) \frac{x_j^2}{r} \, dx = 0$$

for all j. It is easily seen that this integral is independent of j, and so its value is

$$C := \frac{1}{n} \int \frac{\partial F}{\partial a}(a, \Phi_a(r)) \Phi_a'(r) r \, dx$$

which is non-zero by assumption. Therefore, our system reduces to

$$C \sum_i \sigma_i H_{j,i} = 0$$

for all j, and since $H(b)$ is an invertible matrix, this implies $\sigma_0 = 0$, hence our claim.

Now to the final contradiction: we have that $\widetilde{u}_\varepsilon = \nabla \phi_a \cdot \sigma_\varepsilon + \gamma_\varepsilon \widetilde{v}_\varepsilon$ and we now know that for a subsequence $\gamma_\varepsilon \to 0$ and $\sigma_\varepsilon \to 0$. This implies $\widetilde{u}_\varepsilon \to 0$ in H^2, contradicting $\|\widetilde{u}_\varepsilon\|_{H^2} = 1$. This concludes the proof.

We remark that for $c < 2$ it is not the case that $\|T_\varepsilon^{-1}\| = O(\varepsilon^{-c})$. Indeed if $c < 2$ then $\lim_{\varepsilon \to 0} \varepsilon^{-c} T_\varepsilon v_\varepsilon = 0$ in L^2, where $v_\varepsilon(x) = D_j \phi_a(x - \frac{b}{\varepsilon} + s_\varepsilon)$.

4 Remark on the assumptions

The non-vanishing of the integral

$$I := \int \frac{\partial F}{\partial a}(a, \Phi_a(r)) \Phi_a'(r) \, r \, dx \tag{4.1}$$

was used in the proof (as well as in the proof of the very existence of u_ε). This condition seems rather technical and meaningless. To replace it by a more natural one, we shall assume that the ϕ_a form a (smooth) continuum with respect to a, an assumption that we did not need before, since we were always working at fixed a. This is often the case in practice, since the functions ϕ_a are usually obtained by scaling ϕ_1 (for $a = 1$).

Proposition 4.1 *In addition to all previous hypotheses on ϕ_a, assume that it is a C^1 function of a. Then the integral I is non-zero if and only if*

$$\frac{d}{da}\left(\int |\nabla\phi_a|^2\,dx\right) \neq 0$$

Proof. We shall in fact establish the identity

$$I := \int \frac{\partial F}{\partial a}(a, \Phi_a(r))\Phi_a'(r)\,r\,dx = -\frac{d}{da}\left(\int |\nabla\phi_a|^2\,dx\right) \tag{4.2}$$

By spherical symmetry we have

$$\int \frac{\partial F}{\partial a}(a, \Phi_a(r))\Phi_a'(r)\,r\,dx = n\int \frac{\partial F}{\partial a}(a, \phi_a(x))x_j D_j\phi_a(x)\,dx$$

for each j. Differentiating the equation $-\Delta\phi_a + F(a, \phi_a) = 0$ with respect to a gives

$$-\Delta\left(\frac{\partial\phi_a}{\partial a}\right) + \frac{\partial F}{\partial u}(a, \phi_a)\frac{\partial\phi_a}{\partial a} + \frac{\partial F}{\partial a}(a, \phi_a) = 0$$

and therefore

$$\begin{aligned} I &= n\int \left(\Delta - \frac{\partial F}{\partial u}(a, \phi_a)\right)\left(\frac{\partial\phi_a}{\partial a}\right) x_j D_j\phi_a(x)\,dx \\ &= n\int \frac{\partial\phi_a}{\partial a}\left(\Delta - \frac{\partial F}{\partial u}(a, \phi_a)\right)(x_j D_j\phi_a)\,dx \end{aligned}$$

using self-adjointness. We expand

$$\Delta(x_j D_j\phi_a) = x_j\Delta D_j\phi_a + 2D_j^2\phi_a$$

and since the partial derivatives $D_j\phi_a$ belong to the kernel of $-\Delta + \dfrac{\partial F}{\partial u}(a, \phi_a)$, we are left with

$$I = 2n\int \frac{\partial\phi_a}{\partial a}D_j^2\phi_a\,dx$$

This is true for each j, so

$$I = 2\int \frac{\partial\phi_a}{\partial a}\Delta\phi_a\,dx = -2\int \nabla\phi_a\cdot\nabla\left(\frac{\partial\phi_a}{\partial a}\right)dx = -\frac{d}{da}\left(\int |\nabla\phi_a|^2\,dx\right)$$

and the proof is complete.

We now give more details about how a continuum can be obtained from a single solution for a fixed value of a, and how the result of the previous proposition applies then.

Assume we are given a non-trivial, spherically symmetric solution $\psi(x)$ in H^2 of an equation

$$-\Delta u + G(u) = 0$$

where G satisfies our regularity and growth conditions. For a in a bounded interval I, we put

$$\phi_a(x) = a^{\mu}\psi(a^{\nu}x)$$

for positive exponents μ and ν. This is obviously smooth with respect to a. Moreover,

$$\Delta\phi_a = a^{\mu+2\nu}\Delta\psi(a^{\nu}x) = a^{\mu+2\nu}G(a^{-\mu}\phi_a)$$

Therefore ϕ_a solves $-\Delta u + F(a,u) = 0$ where

$$F(a,u) = a^{\mu+2\nu}G(a^{-\mu}u)$$

We see that $F(1,u) = G(u)$. The function F satisfies our growth conditions. According to proposition 4.1, the non-vanishing of the integral 4.1 reduces to

$$0 \neq \frac{d}{da}\left(a^{2\mu+2\nu}\int |\nabla\psi(a^{\nu}x)|^2\,dx\right) = (2\mu+2\nu-n\nu)a^{2\mu+2\nu-n\nu-1}\int |\nabla\psi|^2\,dx$$

So a necessary and sufficient condition for the non-vanishing of this integral is simply

$$2\mu + 2\nu - n\nu \neq 0$$

Under this simple assumption, we can apply Theorems 2.6 and 3.1, using a potential V with range in I.

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Authors' address: The University Science Institute, University of Iceland, Dunhaga 3, 107–Reykjavik, Iceland. E-mail: `robmag@hi.is, oms3@hi.is`.