Integr. equ. oper. theory 99 (9999) 0–0
0378-620X/99000-0 \$ 1.50+0.20/0

Integral Equations and Operator Theory

Natural representations of the multiplicity of an analytic operator-valued function at an isolated point of the spectrum

Robert Magnus and Carlos Mora-Corral

Abstract. Representations are given for the multiplicity of an analytic operator-valued function A at an isolated point z_0 of the spectrum in the form of kernels and ranges of Hankel and Toeplitz matrices whose entries are derived from the Taylor coefficients of A and the Laurent coefficients of A^{-1} about z_0. In two special cases the results can be expressed in terms of finite matrices: when A is a polynomial and when A^{-1} has a pole at z_0. The latter case leads to the theory of Jordan chains.

Mathematics Subject Classification (2000). 47A56.

Keywords. Algebraic multiplicities, isolated points of the spectrum.

1. Multiplicity for analytic operator functions

We begin by recalling one of several constructions of a multiplicity for analytic operator functions $A : \mathcal{D} \to L(E)$ where $\mathcal{D} \subset \mathbb{C}$ is an open set and $L(E)$ denotes the Banach algebra of bounded linear operators on the complex Banach space E. For the background we refer to [1].

The spectrum of A is the set Σ of points $z \in \mathcal{D}$ such that $A(z)$ is not invertible. By an invertible operator in $L(E)$ we shall always mean one whose inverse belongs to $L(E)$. Let $\Omega \subset \mathcal{D}$ be a bounded open set whose closure lies in $\mathcal{D}$ and whose boundary is disjoint with Σ. We say that Ω is admissible for A or that the pair (A, Ω) is admissible. We let $\mathcal{O}(\Omega, E)$ denote the space of all analytic E-valued mappings with domain Ω. The multiplicity $m(A, \Omega)$ of the admissible pair is then the isomorphism class of the Banach space

$$\frac{\mathcal{O}(\Omega, E)}{A\mathcal{O}(\Omega, E)}.$$

If the quotient is finite-dimensional and of dimension d we say that the multiplicity is the finite number d. That the quotient is a Banach space was shown in [1]. This will also become clear in the subsequent paragraphs. We will also find spaces of analytic functions that represent $m(A,\Omega)$.

The key novelty in this definition is its great generality: no Fredholm condition is needed. The following result, proved in [1], collects the main properties of this multiplicity. We quote it for the sake of completeness, though no explicit use of this result will be make in this paper.

Proposition 1.1. *Let E, F be complex Banach spaces, $A : \mathcal{D} \to L(E)$ and $B : \mathcal{D} \to L(F)$ be analytic and Ω be admissible for A and B. Let Σ be the spectrum of A. The following properties are satisfied:*

1. *$m(A,\Omega) = 0$ if and only if $\Omega \cap \Sigma = \varnothing$.*
2. *If $E = F$ then $m(AB,\Omega) = m(A,\Omega) + m(B,\Omega)$. Here the sum of two isomorphism classes of Banach spaces is the isomorphism class of the direct sum of representatives of those spaces.*
3. *$m(A \oplus B,\Omega) = m(A,\Omega) + m(B,\Omega)$.*
4. *If Ω_1 and Ω_2 are admissible for A and $\Omega_1 \cap \Omega_2 = \varnothing$ then $m(A,\Omega_1 \cup \Omega_2) = m(A,\Omega_1) + m(A,\Omega_2)$.*
5. *Let $H : [0,1] \times \mathcal{D} \to L(E)$ be continuous such that Ω is admissible for the analytic maps $H(t,\cdot)$, $0 \le t \le 1$. Then $m(H(0,\cdot),\Omega) = m(H(1,\cdot),\Omega)$.*

The purpose of this paper is to work out the special case of an isolated point of the spectrum. We will see that in that case, the multiplicity can be represented as ranges or kernels of Hankel and Toeplitz matrices. We will also show how the classical concept of Jordan chain (see, e.g. [2, 6]) appears here in a natural way. In particular, it will appear that what lies behind Jordan chains is not the Fredholm condition (as usually assumed in the literature) but simply that the inverse has a pole. So no finite-dimensionality assumptions are needed: the finiteness is only in the pole order. Even so we do not specialize to isolated points until Section 3.

First we introduce some more spaces of analytic operator functions and mappings between them that are induced by an admissible pair (A,Ω). Let $\Sigma_\Omega = \Omega \cap \Sigma$. The space $\mathcal{O}_0(\mathbb{C} \setminus \Sigma_\Omega, E)$ will consist of all analytic E-valued mappings with domain $\mathbb{C} \setminus \Sigma_\Omega$ that vanish at infinity. The spaces $\mathcal{O}(\Omega, E)$ and $\mathcal{O}_0(\mathbb{C} \setminus \Sigma_\Omega, E)$ are Fréchet spaces. We may take as seminorms the suprema of $\|f\|$ on compact subsets of the domain, choosing if necessary countably many corresponding to an increasing sequence of compacta.

We introduce continuous linear mappings

$$\tilde{H} : \mathcal{O}(\Omega, E) \to \mathcal{O}_0(\mathbb{C} \setminus \Sigma_\Omega, E)$$

and

$$H : \mathcal{O}_0(\mathbb{C} \setminus \Sigma_\Omega, E) \to \mathcal{O}(\Omega, E).$$

Of these, $\tilde{H}$ was defined in [1] (but was denoted there by H). We recapitulate the definition. Firstly we recall that an open bounded set $U \subset \mathbb{C}$ is called a Cauchy-domain if its boundary ∂U consists of finitely many disjoint, rectifiable

Jordan-curves. We give the boundary the induced orientation. Let $f \in \mathcal{O}(\Omega, E)$ and $z \in \mathbb{C} \setminus \Sigma_\Omega$. Choose a Cauchy-domain Ω' such that

$$\Sigma_\Omega \subset \Omega' \subset \overline{\Omega'} \subset \Omega \tag{1.1}$$

and $z \notin \overline{\Omega'}$. We set

$$(\tilde{H}f)(z) = \frac{1}{2\pi i} \int_{\partial\Omega'} \frac{A(\zeta)^{-1} f(\zeta)}{z - \zeta}\, d\zeta.$$

We define H as follows. Let $h \in \mathcal{O}_0(\mathbb{C} \setminus \Sigma_\Omega, E)$ and let $z \in \Omega$. Choose a Cauchy-domain Ω' satisfying (1.1) and such that $z \in \Omega'$. We set

$$(Hh)(z) = \frac{1}{2\pi i} \int_{\partial\Omega'} \frac{A(\zeta) h(\zeta)}{\zeta - z}\, d\zeta.$$

The facts set out in the following proposition were mostly derived in [1, Section 2] and will be used throughout the paper. We give detailed proofs only where these were not explicit in [1].

Proposition 1.2.

(i) *The range of $\tilde{H}$ consists of those functions $h \in \mathcal{O}_0(\mathbb{C} \setminus \Sigma_\Omega, E)$ such that Ah may be extended from $\Omega \setminus \Sigma_\Omega$ to an element of $\mathcal{O}(\Omega, E)$. The kernel of $\tilde{H}$ is $A\mathcal{O}(\Omega, E)$.*

(ii) *For each $f \in \mathcal{O}(\Omega, E)$ let Sf be the extension of $A(\tilde{H}f)$ to an element of $\mathcal{O}(\Omega, E)$. Then $S : \mathcal{O}(\Omega, E) \to \mathcal{O}(\Omega, E)$ is a continuous projection.*

(iii) *The kernel of S is $A\mathcal{O}(\Omega, E)$ and its range represents the multiplicity $m(A, \Omega)$.*

(iv) *The range of S consists of those elements $f \in \mathcal{O}(\Omega, E)$ such that $A^{-1}f$ has an analytic extension to $\mathbb{C} \setminus \Sigma_\Omega$ that vanishes at infinity.*

Proof. (i) It was proved in [1] (and it is easy to see, besides) that if $f \in \mathcal{O}(\Omega, E)$, $z \in \Omega \setminus \Sigma_\Omega$, and Ω' is a Cauchy-domain satisfying (1.1), then

$$A(z)(\tilde{H}f)(z) = \frac{1}{2\pi i} \int_{\partial\Omega'} B(z, \zeta) A(\zeta)^{-1} f(\zeta)\, d\zeta, \tag{1.2}$$

where $B : \mathcal{H} \times \mathcal{H} \to L(E)$ is the unique analytic map satisfying

$$A(w) = A(z) + (w - z) B(z, w), \qquad w, z \in \mathcal{H}. \tag{1.3}$$

This proves that $A(\tilde{H}f)$ can be extended to an element of $\mathcal{O}(\Omega, E)$. Conversely, take $h \in \mathcal{O}_0(\mathbb{C} \setminus \Sigma_\Omega, E)$ such that Ah can be extended to an element $f \in \mathcal{O}(\Omega, E)$. Then, for any $z \in \mathbb{C} \setminus \Sigma_\Omega$ and any Cauchy-domain Ω' satisfying (1.1) and such that $z \notin \overline{\Omega'}$, we can apply Cauchy's formula considering $\mathbb{C} \cup \{\infty\} \setminus \overline{\Omega'}$ as the inner domain of $\partial\Omega'$, and deduce

$$(\tilde{H}f)(z) = \frac{1}{2\pi i} \int_{\partial\Omega'} \frac{h(\zeta)}{z - \zeta}\, d\zeta = h(z).$$

Hence h belongs to the range of $\tilde{H}$.

By Cauchy's theorem it is clear that $A\mathcal{O}(\Omega, E)$ is contained in the kernel of $\tilde{H}$. Conversely, if f belongs to the kernel of $\tilde{H}$, then for each $z \in \Omega \setminus \Sigma_\Omega$ and each Cauchy-domain Ω' satisfying (1.1) and such that $z \in \Omega'$, we obtain from (1.2) and (1.3) that

$$f(z) = \frac{A(z)}{2\pi i} \int_{\partial\Omega'} \frac{A(\zeta)^{-1} f(\zeta)}{\zeta - z} \, d\zeta$$

(see [1, Lemma 1] for more details).

Statements (ii)–(iv) are proved in [1, Section 2]. □

We can see why the range of S is a Banach space from the following considerations. Choose a domain Ω' satisfying (1.1). Then $p(f) = \sup_{\partial\Omega'} \|f\|$ is a seminorm of $\mathcal{O}(\Omega, E)$. Let f_n be a sequence in the range of S such that $p(f_n) \to 0$. Then f_n converges uniformly to 0 in Ω' by the maximum principle. Now $A^{-1} f_n$ extends to an element h_n of $\mathcal{O}_0(\mathbb{C} \setminus \Sigma_\Omega, E)$ and converges uniformly to 0 on $\partial\Omega'$. Hence, by the maximum principle, h_n converges uniformly to 0 in $\mathbb{C} \setminus \Omega'$, whence f_n converges uniformly to 0 in Ω. This implies that the topology that $\operatorname{ran} S$ inherits from $\mathcal{O}(\Omega, E)$ is a Banach space topology with norm $p(f)$. In a similar way, the same seminorm is a norm for the range of $\tilde{H}$.

Because the Banach space structure is so transparent, the representation of $m(A, \Omega)$ as the range of S is very convenient. We shall usually have recourse to it rather than the quotient $\mathcal{O}(\Omega, E)/A\mathcal{O}(\Omega, E)$.

We set out some further facts not found in [1].

Proposition 1.3.
(i) $S = H\tilde{H}$.
(ii) $\tilde{H} H \tilde{H} = \tilde{H}$.
(iii) $\tilde{H} H$ *is a projection whose range represents* $m(A, \Omega)$.
(iv) *The range of* $\tilde{H}$ *equals the range of* $\tilde{H} H$.

Proof. (i) Let $f \in \mathcal{O}(\Omega, E)$. It is enough to show that $(H\tilde{H} f)(z) = (Sf)(z)$ for $z \in \Omega \setminus \Sigma_\Omega$. So let $z \in \Omega \setminus \Sigma_\Omega$ and choose Cauchy-domains Ω' and Ω'' such that

$$\Sigma_\Omega \subset \Omega' \subset \overline{\Omega'} \subset \Omega'' \subset \overline{\Omega''} \subset \Omega \tag{1.4}$$

and $z \in \Omega'' \setminus \overline{\Omega'}$. Now we have

$$(H\tilde{H} f)(z) = \left(\frac{1}{2\pi i}\right)^2 \int_{\partial\Omega''} \frac{A(\eta)}{\eta - z} \left(\int_{\partial\Omega'} \frac{A(\zeta)^{-1} f(\zeta)}{\eta - \zeta} \, d\zeta \right) d\eta$$

and inverting the order of the integrals we find by Cauchy's formula

$$\begin{aligned}
(H\tilde{H} f)(z) &= \frac{1}{2\pi i} \int_{\partial\Omega'} \left(\frac{A(z)}{z - \zeta} + \frac{A(\zeta)}{\zeta - z} \right) A(\zeta)^{-1} f(\zeta) \, d\zeta \\
&= \frac{A(z)}{2\pi i} \int_{\partial\Omega'} \frac{A(\zeta)^{-1} f(\zeta)}{z - \zeta} \, d\zeta + \frac{1}{2\pi i} \int_{\partial\Omega'} \frac{f(\zeta)}{\zeta - z} \, d\zeta \\
&= A(z)(\tilde{H} f)(z) \\
&= (Sf)(z).
\end{aligned}$$

(ii) Let $h \in \mathcal{O}_0(\mathbb{C} \setminus \Sigma_\Omega, E)$ be such that Ah has an analytic extension to Ω. Then, as shown in the proof of Proposition 1.2(i), $\tilde{H}Ah = h$. The result now follows from (i).

(iii) That $\tilde{H}H$ is a projection is an obvious consequence of (ii). It also follows that $\tilde{H}$ restricts to an isomorphism of the range of $H\tilde{H}$ onto the range of $\tilde{H}H$. Hence the range of $\tilde{H}H$ represents the multiplicity.

(iv) This follows from (ii). □

Let us summarize the representatives of $m(A, \Omega)$ that we have found.

1. The subspace of $\mathcal{O}(\Omega, E)$ given by the range of the projection $H\tilde{H}$.
2. The subspace of $\mathcal{O}_0(\mathbb{C} \setminus \Sigma_\Omega, E)$ given by the range of the projection $\tilde{H}H$, which equals the range of $\tilde{H}$.

Note that we do not have a perfect symmetry between H and $\tilde{H}$. In general the range of H can be larger than the range of $H\tilde{H}$ and does not represent the multiplicity. An example of this will be given at the end of Section 3.

The multiplicity $m(A, \Omega)$ depends only on A and Σ_Ω. For, as shown in [1], if Ω_1 is admissible and $\Sigma_{\Omega_1} = \Sigma_\Omega$ then $m(A, \Omega_1) = m(A, \Omega)$. It was shown in [1] that the restriction map $\mathcal{O}(\Omega, E) \to \mathcal{O}(\Omega \cap \Omega_1, E)$ induces an isomorphism $\mathcal{O}(\Omega, E)/A\mathcal{O}(\Omega, E) \to \mathcal{O}(\Omega \cap \Omega_1, E)/A\mathcal{O}(\Omega \cap \Omega_1, E)$. We can see this from the characterization of the range of $\tilde{H}$ given by Proposition 1.2(i). If $h \in \mathcal{O}_0(\mathbb{C} \setminus \Sigma_\Omega, E)$ is such that Ah may be extended analytically from $\Omega \setminus \Sigma_\Omega$ to Ω then it can clearly also be extended analytically from $\Omega_1 \setminus \Sigma_\Omega$ to Ω_1.

The upshot of the last paragraph is that we can replace the space $\mathcal{O}(\Omega, E)$ by the space $\mathcal{O}(\Sigma_\Omega, E)$ of E-valued functions analytic in some neighbourhood of the closed set Σ_Ω. We can define the mappings H and $\tilde{H}$ much as before, tailoring the Cauchy-domain Ω' to the function f in the definition of $\tilde{H}$. The range of $H\tilde{H}$ is really just the same as before, up to analytic continuation of its elements.

The space $\mathcal{O}(\Sigma_\Omega, E)$ may be topologized as the inductive limit of the spaces $\mathcal{O}(\Omega, E)$ as Ω ranges over open sets containing Σ_Ω, or, more conveniently but equivalently, as the inductive limit of the Banach spaces $\mathcal{H}(\bar{\Omega}, E)$ of E-valued functions analytic in Ω and continuous in $\bar{\Omega}$, as Ω ranges over open bounded sets containing Σ_Ω.

2. The multiplicity as a kernel

Having represented $m(A, \Omega)$ by ranges we can ask whether we can represent it by kernels. There are of course the obvious ones: the kernels of $I - \tilde{H}H$ and $I - H\tilde{H}$. We now show that two operators J and $\tilde{J}$ can be defined in a way very similar to that used for H and $\tilde{H}$, such that their kernels represent the multiplicity. Explicitly, we introduce

$$J : \mathcal{O}_0(\mathbb{C} \setminus \Sigma_\Omega, E) \to \mathcal{O}_0(\mathbb{C} \setminus \Sigma_\Omega, E)$$

and

$$\tilde{J} : \mathcal{O}(\Omega, E) \to \mathcal{O}(\Omega, E)$$

as follows. Let $h \in \mathcal{O}_0(\mathbb{C} \setminus \Sigma_\Omega, E)$ and $z \in \mathbb{C} \setminus \Sigma_\Omega$. Choose a Cauchy-domain Ω' satisfying (1.1) and such that $z \notin \overline{\Omega'}$. We set

$$(Jh)(z) = \frac{1}{2\pi i} \int_{\partial\Omega'} \frac{A(\zeta)h(\zeta)}{z-\zeta}\, d\zeta.$$

We define $\tilde{J}$ as follows. Let $f \in \mathcal{O}(\Omega, E)$ and let $z \in \Omega$. Choose a Cauchy-domain Ω' satisfying (1.1) and such that $z \in \Omega'$. We set

$$(\tilde{J}f)(z) = \frac{1}{2\pi i} \int_{\partial\Omega'} \frac{A(\zeta)^{-1}f(\zeta)}{\zeta - z}\, d\zeta.$$

Theorem 2.1. *The kernel of J equals the range of $\tilde{H}$. The kernel of $\tilde{J}$ equals the range of S.*

Proof. If $h \in \operatorname{ran} \tilde{H}$ then Ah has an analytic extension to Ω; hence $Jh = 0$. Conversely, suppose that h is in the kernel of J. Let $z \in \Omega \setminus \Sigma_\Omega$ and consider Cauchy-domains Ω', Ω'' satisfying (1.4) and such that $z \in \Omega'' \setminus \overline{\Omega'}$. Then, denoting by $C_\varepsilon(z)$ the circumference of sufficiently small radius $\varepsilon > 0$ and centre z, positively oriented, we have

$$\begin{aligned}
(Hh)(z) &= \frac{1}{2\pi i} \int_{\partial\Omega''} \frac{A(\zeta)h(\zeta)}{\zeta - z}\, d\zeta \\
&= \frac{1}{2\pi i} \int_{\partial\Omega'} \frac{A(\zeta)h(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int_{C_\varepsilon(z)} \frac{A(\zeta)h(\zeta)}{\zeta - z}\, d\zeta \\
&= A(z)h(z).
\end{aligned}$$

This proves that Ah has an analytic extension to Ω, hence showing that h is in the range of $\tilde{H}$.

If $f \in \operatorname{ran} S$ then $A^{-1}f$ has an analytic extension to $\mathbb{C} \setminus \Sigma_\Omega$ vanishing at infinity; hence $\tilde{J}f = 0$. Conversely, suppose that f is in the kernel of $\tilde{J}$. Let $z \in \mathbb{C} \setminus \Sigma_\Omega$ and consider Cauchy-domains Ω', Ω'' satisfying (1.4) and such that $z \in \Omega'' \setminus \overline{\Omega'}$. Then, for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned}
(\tilde{H}f)(z) &= \frac{1}{2\pi i} \int_{\partial\Omega'} \frac{A(\zeta)^{-1}f(\zeta)}{z - \zeta}\, d\zeta \\
&= \frac{1}{2\pi i} \int_{\partial\Omega''} \frac{A(\zeta)^{-1}f(\zeta)}{z - \zeta}\, d\zeta - \frac{1}{2\pi i} \int_{C_\varepsilon(z)} \frac{A(\zeta)^{-1}f(\zeta)}{z - \zeta}\, d\zeta \\
&= A(z)^{-1}f(z).
\end{aligned}$$

This proves that $A^{-1}f$ has an analytic extension to $\mathbb{C} \setminus \Sigma_\Omega$ vanishing at infinity, hence showing that f is in the range of S. □

3. Isolated points of the spectrum

In the case when Σ_Ω is a single point, say $\Sigma_\Omega = \{z_0\}$, all the representations of $m(A, \Omega)$ that we have obtained so far can be expressed, in a natural way, as ranges and kernels of infinite matrices with entries in $L(E)$. In this case, $m(A,\Omega)$ is called the multiplicity of A at the point z_0, because, as shown in [1], it only depends on A and z_0. Let us denote the space of E-valued principal parts at z_0 by $\mathcal{P}(\{z_0\},E)$. Now the space $\mathcal{O}_0(\mathbb{C}\setminus\Sigma_\Omega,E)$ is identified with $\mathcal{P}(\{z_0\},E)$. Also, $\mathcal{O}(\{z_0\},E)$ stands for the space of all E-valued analytic parts at z_0.

It will be convenient to represent the linear operators H, $\tilde{H}$, J and $\tilde{J}$ by infinite matrices. We suppose that $A(z)$ has the Taylor expansion

$$A(z)=\sum_{k=0}^{\infty}(z-z_0)^k A_k$$

convergent in some disc centred at z_0, whilst $A(z)^{-1}$ has a Laurent expansion

$$A(z)^{-1}=\sum_{k=-\infty}^{\infty}(z-z_0)^k \tilde{A}_k$$

convergent in some punctured disc centred at z_0. The coefficients A_k and $\tilde{A}_k$ are elements of $L(E)$.

We can identify $\mathcal{O}(\{z_0\},E)$ with the space of all sequences $(u_k)_{k=0}^{\infty}$ in E such that the series $\sum(z-z_0)^k u_k$ has positive radius of convergence. Similarly, $\mathcal{P}(\{z_0\},E)$ can be identified with the space of all sequences $(u_k)_{k=1}^{\infty}$ in E such that $\sum(z-z_0)^k u_k$ is convergent for all z; this corresponds to the principal part $\sum_{k=1}^{\infty}(z-z_0)^{-k}u_k$. The following result provides us with two descriptions of each of the operators H, $\tilde{H}$, J and $\tilde{J}$, the second one being in each case an infinite matrix with entries in $L(E)$.

Theorem 3.1. *Using the identifications above, we have the following descriptions:*

(i) $H:\mathcal{P}(\{z_0\},E)\to\mathcal{O}(\{z_0\},E)$ *is given by*

$$Hh = \textit{analytic part of } Ah \textit{ at } z_0$$

and has the matrix

$$\mathcal{A}=\begin{pmatrix} A_1 & A_2 & A_3 & \cdots \\ A_2 & A_3 & A_4 & \cdots \\ A_3 & A_4 & A_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

(ii) $\tilde{H}:\mathcal{O}(\{z_0\},E)\to\mathcal{P}(\{z_0\},E)$ *is given by*

$$\tilde{H}f = \textit{principal part of } A^{-1}f \textit{ at } z_0$$

and has the matrix

$$\tilde{\mathcal{A}}=\begin{pmatrix} \tilde{A}_{-1} & \tilde{A}_{-2} & \tilde{A}_{-3} & \cdots \\ \tilde{A}_{-2} & \tilde{A}_{-3} & \tilde{A}_{-4} & \cdots \\ \tilde{A}_{-3} & \tilde{A}_{-4} & \tilde{A}_{-5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

(iii) $J : \mathcal{P}(\{z_0\}, E) \to \mathcal{P}(\{z_0\}, E)$ *is given by*

$$Jh = \textit{principal part of } Ah \textit{ at } z_0$$

and has the matrix

$$\mathcal{A}^{\Delta} = \begin{pmatrix} A_0 & A_1 & A_2 & \cdots \\ & A_0 & A_1 & \cdots \\ & & A_0 & \cdots \\ & & & \ddots \end{pmatrix}.$$

(iv) $\tilde{J} : \mathcal{O}(\{z_0\}, E) \to \mathcal{O}(\{z_0\}, E)$ *is given by*

$$\tilde{J}f = \textit{analytic part of } A^{-1}f \textit{ at } z_0$$

and has the matrix

$$\tilde{\mathcal{A}}_0 = \begin{pmatrix} \tilde{A}_0 & \tilde{A}_{-1} & \tilde{A}_{-2} & \cdots \\ \tilde{A}_1 & \tilde{A}_0 & \tilde{A}_{-1} & \cdots \\ \tilde{A}_2 & \tilde{A}_1 & \tilde{A}_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proof. (i) Let $h \in \mathcal{P}(\{z_0\}, E)$ and let

$$(Ah)(\zeta) = \sum_{n=-\infty}^{\infty} (\zeta - z_0)^n B_n$$

be the Laurent expansion of Ah in some punctured disc centred at z_0. Let Ω be a small disc centred at z_0. Then for any $z \in \Omega$, we may apply Cauchy's Theorem, considering $\mathbb{C} \cup \{\infty\} \setminus \bar{\Omega}$ as the inner domain of $\partial\Omega$, and obtain

$$\int_{\partial\Omega} \frac{1}{\zeta - z} \sum_{n=-\infty}^{-1} (\zeta - z_0)^n B_n \, d\zeta = 0.$$

Therefore,

$$Hh(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{1}{\zeta - z} \sum_{n=0}^{\infty} (\zeta - z_0)^n B_n \, d\zeta = \sum_{n=0}^{\infty} (z - z_0)^n B_n,$$

which equals the analytic part of Ah at z_0. Furthermore, if $h = \sum_{j=1}^{\infty} (z - z_0)^{-j} u_j$ then $B_n = \sum_{k=1}^{\infty} A_{n+k} u_k$, $n \geq 0$, thus giving the stated matrix.

(ii) Let $f \in \mathcal{O}(\{z_0\}, E)$ and

$$(A^{-1}f)(\zeta) = \sum_{n=-\infty}^{\infty} (\zeta - z_0)^n B_n$$

be the Laurent expansion of $A^{-1}f$ in some punctured disc centred at z_0. Let Ω be a small disc centred at z_0. Then, for any $z \notin \bar{\Omega}$, we have

$$\int_{\partial\Omega} \frac{1}{z - \zeta} \sum_{n=0}^{\infty} (\zeta - z_0)^n B_n \, d\zeta = 0,$$

by Cauchy's formula. Therefore, by Cauchy's formula, considering $\mathbb{C}\cup\{\infty\}\setminus\bar{\Omega}$ as the inner domain of $\partial\Omega$,

$$\tilde{H}f(z)=\frac{1}{2\pi i}\int_{\partial\Omega}\frac{1}{z-\zeta}\sum_{n=-\infty}^{-1}(\zeta-z_0)^n B_n\,d\zeta=\sum_{n=-\infty}^{-1}(z-z_0)^n B_n,$$

which equals the principal part of $A^{-1}f$ at z_0. Furthermore, if $f=\sum_{j=0}^{\infty}(z-z_0)^j u_j$ then $B_n=\sum_{k=0}^{\infty}\tilde{A}_{-n-k}u_k$, $n\le -1$.

Parts (iii) and (iv) of the Theorem are proved analogously. □

Theorem 3.1 is not surprising at all. These formulas are familiar whenever one represents a bounded multiplication operator in spaces with a basis; for example, in spaces of functions with Fourier series, Taylor series or Laurent series, or with the canonical basis in ℓ_p, etc. Matrices of the form appearing in (i) and (ii) of Theorem 3.1 are called Hankel matrices, while those of the form in (iii) and (iv) are called Toeplitz matrices. See [11] or [10] for these ideas. The key point here is that, thanks to Propositions 1.2, 1.3 and Theorem 2.1, the kernels or ranges of those matrices represent the multiplicity.

With these representations, we are able to give an example of a function A such that the range of H does not represent the multiplicity. Consider the function $A:\mathbb{C}\to L(\mathbb{C}^2)$ defined by the diagonal matrix $A(z)=\operatorname{diag}\{1+z,z\}$, and let Ω be the disc of radius $1/2$ centred at 0. Then, $\Sigma_\Omega=\{0\}$ and A^{-1} has a pole of order 1 at 0. Following the notations above for the coefficients we have $A_1=\operatorname{diag}\{1,1\}$, $A_n=0$ for $n\ge 2$, $\tilde{A}_{-1}=\operatorname{diag}\{0,1\}$, $\tilde{A}_n=0$ for $n\le -2$. Then, the dimension of the range of the matrix $\mathcal{A}$ is 2, whereas the dimension of the range of the matrix $\mathcal{A}\tilde{\mathcal{A}}$ is 1.

4. The canonical linearization in the range of $\tilde{H}$

We begin by recalling the concept of linearization (see, e.g., Gohberg et al. [5]). A linearization of A in the admissible set Ω is an operator T defined on a Banach space F such that there exist Banach spaces X, Y and analytic operator-functions $\Phi:\Omega\to L(F\oplus Y,E\oplus X)$, $\Psi:\Omega\to L(E\oplus X,F\oplus Y)$, such that Φ and Ψ take invertible values only, and

$$A(z)\oplus I_X=\Phi(z)\,[(zI_F-T)\oplus I_Y]\,\Psi(z),\quad z\in\Omega.$$

It was shown in [5] that linearizations always exist. In [1, Section 6] a particular linearization of A in Ω was constructed which was called *canonical*. It was given by the linear map

$$\bar{\omega}:\frac{\mathcal{O}(\Omega,E)}{A\mathcal{O}(\Omega,E)}\to\frac{\mathcal{O}(\Omega,E)}{A\mathcal{O}(\Omega,E)}$$

induced by $\omega : \mathcal{O}(\Omega, E) \to \mathcal{O}(\Omega, E)$, where $(\omega f)(z) = zf(z)$; or, in brief, $\bar{\omega}$ is induced by multiplication by z. We may view $\bar{\omega}$ as the map that makes the diagram

$$\begin{array}{ccc} \mathcal{O}(\Omega,E) & \xrightarrow{\ \omega\ } & \mathcal{O}(\Omega,E) \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ \frac{\mathcal{O}(\Omega,E)}{A\mathcal{O}(\Omega,E)} & \xrightarrow{\ \bar{\omega}\ } & \frac{\mathcal{O}(\Omega,E)}{A\mathcal{O}(\Omega,E)} \end{array}$$

commute, where π is the canonical surjection. The spectrum of the canonical linearization was shown to be the set Σ_Ω (see [1, Section 6]).

Given any representation F of $m(A, \Omega)$ together with an isomorphism $\alpha : \frac{\mathcal{O}(\Omega,E)}{A\mathcal{O}(\Omega,E)} \to F$ there is a representation τ of $\bar{\omega}$ in F; namely τ is the map that makes the diagram

$$\begin{array}{ccc} \frac{\mathcal{O}(\Omega,E)}{A\mathcal{O}(\Omega,E)} & \xrightarrow{\ \bar{\omega}\ } & \frac{\mathcal{O}(\Omega,E)}{A\mathcal{O}(\Omega,E)} \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\alpha}\downarrow \\ F & \xrightarrow{\ \tau\ } & F \end{array}$$

commute.

In particular, this is the case when F is the range of $\tilde{H}$. Since the kernel of $\tilde{H}$ is $A\mathcal{O}(\Omega, E)$ we get an induced isomorphism $\bar{\tilde{H}} : \mathcal{O}(\Omega, E)/A\mathcal{O}(\Omega, E) \to \operatorname{ran} \tilde{H}$ characterized by $\tilde{H} = \bar{\tilde{H}}\pi$.

Proposition 4.1. *The canonical linearization is represented in the range of $\tilde{H}$ by the mapping χ where*

$$(\chi h)(z) = zh(z) - \lim_{z\to\infty} zh(z).$$

In particular, if $\Sigma_\Omega = \{z_0\}$ then χ is given by

$$(\chi h)(z) = \sum_{n=1}^{\infty} (z - z_0)^{-n} u_{n+1} + z_0 h(z)$$

where $h(z) = \sum_{n=1}^{\infty} (z - z_0)^{-n} u_n$. In other words, χ is the left shift plus z_0 times the identity operator on sequences $(u_n)_{n=1}^{\infty}$.

Proof. The second part follows from the first. To prove the first formula we first show that $\tilde{H}\omega = \chi\tilde{H}$. Let $f \in \mathcal{O}(\Omega, E)$. Choosing the Cauchy-domain Ω' so that

(1.1) holds and $z \notin \overline{\Omega'}$, we have

$$\begin{aligned}
(\tilde{H}\omega f)(z) &= \frac{1}{2\pi i}\int_{\partial\Omega'} \frac{A(\zeta)^{-1}\zeta f(\zeta)}{z-\zeta}\,d\zeta \\
&= \frac{1}{2\pi i}\int_{\partial\Omega'} \left(\frac{z}{z-\zeta}-1\right) A(\zeta)^{-1} f(\zeta)\,d\zeta \\
&= z(\tilde{H}f)(z) - \frac{1}{2\pi i}\int_{\partial\Omega'} A(\zeta)^{-1} f(\zeta)\,d\zeta \\
&= z(\tilde{H}f)(z) - \lim_{z\to\infty} z(\tilde{H}f)(z) \\
&= (\chi\tilde{H}f)(z).
\end{aligned}$$

Now we have

$$\chi\bar{\tilde{H}}\pi = \chi\tilde{H} = \tilde{H}\omega = \bar{\tilde{H}}\pi\omega = \bar{\tilde{H}}\bar{\omega}\pi.$$

Since π is surjective we deduce that $\chi\bar{\tilde{H}} = \bar{\tilde{H}}\bar{\omega}$. □

5. The case when A^{-1} has a pole

Our object in this section is to find representations of the multiplicity as complemented subspaces of E^N, for some finite N, using finite matrices whose entries are derived algebraically from the coefficients A_n and $\tilde{A}_n$. In particular we shall derive the Jordan chains of Markus and Sigal [9] (see also [6, 2]).

Proposition 5.1. *Suppose that $\Sigma_\Omega = \{z_0\}$ and that A^{-1} has a pole of order N at z_0. Then the range of $\tilde{H}$ consists only of poles of order at most N, and, in fact, it contains at least one element with pole order precisely N.*

Proof. Let h be in the range of $\tilde{H}$. Then Ah extends to an analytic function f in a neighbourhood of z_0. Then $h = A^{-1}f$, which plainly has a pole of order at most N. This proves the first claim (it also follows from Theorem 3.1(ii)).

We now observe that the canonical linearization χ in the range of $\tilde{H}$ of Proposition 4.1 satisfies $(\chi - z_0 I)^N = 0$. Moreover, by the definition of linearization, we find that A^{-1} and $(zI-\chi)^{-1}$ have the same pole order at z_0, namely N. But then $(\chi - z_0 I)^{N-1} \neq 0$. Hence the range of $\tilde{H}$ must contain at least one genuine pole of order N. □

Note that we have the following converse of Proposition 5.1: if $\Sigma_\Omega = \{z_0\}$ and the range of $\tilde{H}$ consists only of poles of order at most N, then A^{-1} has a pole of order at most N at z_0. This follows at once from Theorem 3.1(ii).

We now apply Theorems 2.1 and 3.1(iii) to the conclusion of Proposition 5.1. We find that the calculation of the representation $\operatorname{ran}\tilde{H}$ in the case when A^{-1} has a pole of order N at z_0 reduces to solving the finite set of equations with upper triangular matrix

$$\begin{pmatrix} A_0 & A_1 & \cdots & A_{N-1} \\ & A_0 & \cdots & A_{N-2} \\ & & \ddots & \vdots \\ & & & A_0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} = 0.$$

Subsidiary to this equation set is a sequence of equation sets

$$\begin{pmatrix} A_0 & A_1 & \cdots & A_{k-1} \\ & A_0 & \cdots & A_{k-2} \\ & & \ddots & \vdots \\ & & & A_0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{pmatrix} = 0. \tag{5.1}$$

For each $k \geq 1$ denote by D_k the solution space of (5.1). It is clear that $D_k \times \{0\} \subset D_{k+1}$ for every $k \geq 1$. Moreover, for $1 \leq k \leq N-1$ this inclusion is strict; indeed, suppose that for some $k \in \{1, \ldots, N-1\}$, every $(u_1, \ldots, u_{k+1})$ in D_{k+1} satisfies $u_{k+1} = 0$. Then, for every $(u_1, \ldots, u_N)$ in D_N, $(u_{N-k}, \ldots, u_N)$ belongs to D_{k+1}, and hence, $u_N = 0$, contradicting Proposition 5.1. Also, if $(u_1, \ldots, u_k)$ is in D_k for some $k > N$, we find by Proposition 5.1 that $u_j = 0$ for $j > N$.

Normally equations (5.1) are presented with the sequences reversed. Let $v_j = u_{k-j}$, $0 \leq j \leq k-1$. Then we have

$$\begin{pmatrix} A_0 & & & \\ A_1 & A_0 & & \\ \vdots & \vdots & \ddots & \\ A_{k-1} & A_{k-2} & \cdots & A_0 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{k-1} \end{pmatrix} = 0. \tag{5.2}$$

We have here the equations giving the Jordan chains of Markus and Sigal [9] (see also [6, 2]), used by them to define the multiplicity in the case when the operator function A takes its values within the class of Fredholm operators of index 0. We do not have any restriction to Fredholm operators; we suppose only that A^{-1} has a pole at z_0.

Let M_k denote the solution space of (5.2). We embed M_k into M_{k+1} by

$$M_k \ni (v_0, \ldots, v_{k-1}) \mapsto \sigma(v_0, \ldots, v_{k-1}) = (0, v_0, \ldots, v_{k-1}) \in M_{k+1}.$$

If A^{-1} has a pole of order N then we have shown that $\sigma(M_k)$ is a proper subspace of M_{k+1} for $k = 1, \ldots, N-1$. Thereafter we have equality and the "stabilized" solution space represents the multiplicity. If we only know that A^{-1} has a pole of some order, then we can compute the multiplicity by calculating M_k until it stabilizes in the sense just explained. However, if A^{-1} has an essential singularity this process may not work. For example if $A(z) = zI - T$, where T is injective, and $z_0 = 0$, then the process stabilizes at once with all solutions zero; however, z_0 could still be an isolated point of the spectrum, in which case the multiplicity is non-zero. In such cases we have to use Theorem 3.1 and represent the multiplicity as the nullspace of the infinite matrix $\mathcal{A}^{\Delta}$, or, alternatively, as the nullspace of the infinite matrix $\tilde{\mathcal{A}}_0$.

The case of Fredholm-operator-valued A is a little different. Standard theory (e.g., Gohberg and Sigal [7]) shows that for such functions, even without assuming that the point z_0 is isolated in the spectrum, the following are equivalent:

1. The above process stabilizes.
2. The point z_0 is isolated in the spectrum.
3. The point z_0 is a pole of A^{-1}.
4. The multiplicity is a finite number.

There is another interesting difference. In general A^{-1} has a pole at z_0 of order less than or equal to ν if and only if $\|A^{-1}(z)\| = O(|z - z_0|^{-\nu})$ as $z \to z_0$. In the case when A is Fredholm-valued it is sufficient if this holds for real $z - z_0$; this is apparent from the characterization of algebraic eigenvalues given in [8]. This is false if A is not Fredholm-valued. A counterexample is furnished by the operator-function $A(z) = izI_E - T$ where $E = \mathcal{C}([0,1], \mathbb{C})$ and $T \in L(E)$ is the operator

$$Tf(t) = \int_0^t f(s)\,ds\,, \quad t \in [0,1]\,,\ f \in E\,.$$

Then, $A(z)$ is invertible for $z \in \mathbb{C} \setminus \{0\}$ and

$$(A(z)^{-1}f)(t) = \frac{1}{iz}f(t) - \frac{1}{z^2}\int_0^t f(s)e^{(t-s)/(iz)}\,ds\,, \quad t \in [0,1]\,,\ f \in E$$

and, hence, $\|A(z)^{-1}\| \le 2|z|^{-2}$ for *real* $0 < |z| \le 1$, whilst A^{-1} has an essential singularity at $z = 0$.

Let us summarize what seems most useful in the above discussion together with some further consequences of Theorem 3.1.

Theorem 5.2. *Let A^{-1} have a pole of order N at z_0. The following conclusions then hold:*

(i) *The solution space of the triangular system of equations*

$$\begin{pmatrix} A_0 & & \\ \vdots & \ddots & \\ A_{N-1} & \cdots & A_0 \end{pmatrix} \begin{pmatrix} v_0 \\ \vdots \\ v_{N-1} \end{pmatrix} = 0$$

represents the multiplicity as a complemented subspace of E^N.

(ii) *The solution space of the system in* (i) *is the range of the matrix*

$$\begin{pmatrix} \tilde{A}_{-N} & & \\ \vdots & \ddots & \\ \tilde{A}_{-1} & \cdots & \tilde{A}_{-N} \end{pmatrix}.$$

(iii) *The solution space of the system in* (i) *is the range of the projection matrix*

$$\begin{pmatrix} \tilde{A}_{-N} & & \\ \vdots & \ddots & \\ \tilde{A}_{-1} & \cdots & \tilde{A}_{-N} \end{pmatrix} \begin{pmatrix} A_N & \cdots & A_1 \\ \vdots & \ddots & \vdots \\ A_{2N-1} & \cdots & A_N \end{pmatrix}.$$

(iv) *The range of the projection matrix*

$$\begin{pmatrix} A_N & \cdots & A_1 \\ \vdots & \ddots & \vdots \\ A_{2N-1} & \cdots & A_N \end{pmatrix} \begin{pmatrix} \tilde{A}_{-N} & & \\ \vdots & \ddots & \\ \tilde{A}_{-1} & \cdots & \tilde{A}_{-N} \end{pmatrix} \tag{5.3}$$

represents the multiplicity as a complemented subspace of E^N.

Proof. (i) See the above discussion.

(ii) By Theorem 2.1, the kernel of J equals the range of $\tilde{H}$, or, in matrix form (see Theorem 3.1), $\ker \mathcal{A}^{\Delta} = \operatorname{ran} \tilde{\mathcal{A}}$. Since $\tilde{A}_{-k} = 0$ for $k > N$, this gives

$$\ker \begin{pmatrix} A_0 & \cdots & A_{N-1} \\ & \ddots & \vdots \\ & & A_0 \end{pmatrix} = \operatorname{ran} \begin{pmatrix} \tilde{A}_{-1} & \cdots & \tilde{A}_{-N} \\ \vdots & \cdot^{\cdot^{\cdot}} & \\ \tilde{A}_{-N} & & \end{pmatrix},$$

and, reversing the sequences, we obtain (ii).

(iii) Let us call

$$B := \begin{pmatrix} \tilde{A}_{-1} & \cdots & \tilde{A}_{-N} \\ \vdots & \cdot^{\cdot^{\cdot}} & \\ \tilde{A}_{-N} & & \end{pmatrix}, \quad C := \begin{pmatrix} A_1 & \cdots & A_N \\ \vdots & \ddots & \vdots \\ A_N & \cdots & A_{2N-1} \end{pmatrix}.$$

Since $\tilde{H}H$ is a projection with the same range as $\tilde{H}$ we conclude, by Theorem 3.1 and the fact that $\tilde{A}_{-k} = 0$ for $k > N$, that BC is a projection with the same range as B. Then,

$$\begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} BC \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$$

is a projection with the same range as

$$\begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} B.$$

The result now follows from (ii).

(iv) The fact that $H\tilde{H}$ is a projection implies, by Theorem 3.1 and $\tilde{A}_{-k} = 0$ for $k > N$, that CB is a projection. Note that CB equals (5.3). As BC and CB are projections, we conclude that B is an isomorphism from $\operatorname{ran}(CB)$ to $\operatorname{ran}(BC)$. □

Note that the representation of the multiplicity described in three different ways in (i), (ii) and (iii) is the range of $\tilde{H}$ and therefore a subspace of $\mathcal{P}(\{z_0\}, E)$, which, under the conditions of the theorem, consists of poles of order less than or equal to N. The representation of (iv) is quite different, being a subspace of $\mathcal{O}(\{z_0\}, E)$ that consists of polynomials of degree less than or equal to $N-1$. It is not, however, the range of $H\tilde{H}$ in general, but consists of the curtailment of the elements of the range of $H\tilde{H}$ to polynomials of degree less than or equal to $N-1$.

Now we present two sufficient conditions for the multiplicity to be finite.

Proposition 5.3. *Let A^{-1} have a pole of order N at z_0. If $\tilde{A}_{-1}, \dots, \tilde{A}_{-N}$ are compact or $A_1, \dots, A_N$ are compact then the multiplicity of A at z_0 is finite.*

Proof. Suppose $\tilde{A}_{-1}, \dots, \tilde{A}_{-N}$ are compact. Then, by Theorem 5.2(iii),

$$\begin{pmatrix} \tilde{A}_{-N} & & \\ \vdots & \ddots & \\ \tilde{A}_{-1} & \cdots & \tilde{A}_{-N} \end{pmatrix} \begin{pmatrix} A_N & \cdots & A_1 \\ \vdots & \ddots & \vdots \\ A_{2N-1} & \cdots & A_N \end{pmatrix}$$

is a compact projection, hence of finite rank.

Now suppose that $A_1, \dots, A_N$ are compact. We define the functions $B(z) = \sum_{n=0}^{N} (z - z_0)^n A_n$, and $F(z) = B(z)A(z)^{-1}$, the latter being defined in a neighbourhood of z_0. We have

$$F(z) = \left[A(z) + O((z - z_0)^{N+1})\right] A(z)^{-1} = I + O(z - z_0),$$

showing that F is analytic and invertible in a neighbourhood of z_0. Hence A and B are equivalent functions in a neighbourhood of z_0, and, by [1], they have the same multiplicity at z_0. Writing the Laurent expansion

$$B(z)^{-1} = \sum_{n=-N}^{\infty} (z - z_0)^n \tilde{B}_n$$

of B^{-1} at z_0, we have by Theorem 5.2(iii) that a representation of the multiplicity is given by the range of the projection matrix

$$\begin{pmatrix} \tilde{B}_{-N} & & \\ \vdots & \ddots & \\ \tilde{B}_{-1} & \cdots & \tilde{B}_{-N} \end{pmatrix} \begin{pmatrix} A_N & \cdots & A_1 \\ & \ddots & \vdots \\ & & A_N \end{pmatrix},$$

which is compact, and hence of finite rank. □

Remark 5.4. The rank of a projection of finite rank equals its trace. Assume that A^{-1} has a pole of order N at z_0 and $\tilde{A}_{-1}, \dots, \tilde{A}_{-N}$ are compact. Then, as shown in the proof of Proposition 5.3, the multiplicity of A at z_0 equals the trace of the product

$$\begin{pmatrix} \tilde{A}_{-N} & & \\ \vdots & \ddots & \\ \tilde{A}_{-1} & \cdots & \tilde{A}_{-N} \end{pmatrix} \begin{pmatrix} A_N & \cdots & A_1 \\ \vdots & \ddots & \vdots \\ A_{2N-1} & \cdots & A_N \end{pmatrix}.$$

The outcome is

$$\operatorname{tr} \left(\sum_{n=1}^{N} n \tilde{A}_{-n} A_n \right) = \operatorname{tr} \left(\frac{1}{2\pi i} \int_C A(z)^{-1} A'(z) \, dz \right),$$

where C is a small positively oriented circle enclosing z_0. This is the Principle of the Argument which was established in Gohberg and Sigal [7] (see also [4]) under the assumption that $A(z)$ takes its values in the class of Fredholm operators of index 0. Analogously, the multiplicity of A at z_0 equals the trace of the projection

$$\begin{pmatrix} A_N & \cdots & A_1 \\ \vdots & \ddots & \vdots \\ A_{2N-1} & \cdots & A_N \end{pmatrix} \begin{pmatrix} \tilde{A}_{-N} & & \\ \vdots & \ddots & \\ \tilde{A}_{-1} & \cdots & \tilde{A}_{-N} \end{pmatrix},$$

and the outcome is

$$\operatorname{tr}\left(\sum_{n=1}^{N} nA_n\tilde{A}_{-n}\right) = \operatorname{tr}\left(\frac{1}{2\pi i}\int_C A'(z)A(z)^{-1}\,dz\right).$$

6. The case when A is a polynomial

The case when A is a polynomial does not offer such a rich choice of representations of the multiplicity as a subspace of E^N using finite matrices as does the case when A^{-1} has a pole.

Theorem 6.1. *Let A be a polynomial of degree N and let z_0 be an isolated point of the spectrum. Then the multiplicity may be represented as a complemented subspace of E^N in two ways. Firstly, by the range of the projection matrix*

$$\begin{pmatrix} A_1 & \cdots & A_N \\ \vdots & . \cdot \cdot & \\ A_N & & \end{pmatrix} \begin{pmatrix} \tilde{A}_{-1} & \cdots & \tilde{A}_{-N} \\ \vdots & \ddots & \vdots \\ \tilde{A}_{-N} & \cdots & \tilde{A}_{-2N+1} \end{pmatrix}.$$

Secondly, by the range of the projection matrix

$$\begin{pmatrix} \tilde{A}_{-1} & \cdots & \tilde{A}_{-N} \\ \vdots & \ddots & \vdots \\ \tilde{A}_{-N} & \cdots & \tilde{A}_{-2N+1} \end{pmatrix} \begin{pmatrix} A_1 & \cdots & A_N \\ \vdots & . \cdot \cdot & \\ A_N & & \end{pmatrix},$$

which equals the range of

$$\begin{pmatrix} \tilde{A}_{-1} & \cdots & \tilde{A}_{-N} \\ \vdots & \ddots & \vdots \\ \tilde{A}_{-N} & \cdots & \tilde{A}_{-2N+1} \end{pmatrix}.$$

Proof. First, let us call

$$B := \begin{pmatrix} A_1 & \cdots & A_N \\ \vdots & . \cdot \cdot & \\ A_N & & \end{pmatrix}, \quad C := \begin{pmatrix} \tilde{A}_{-1} & \cdots & \tilde{A}_{-N} \\ \vdots & \ddots & \vdots \\ \tilde{A}_{-N} & \cdots & \tilde{A}_{-2N+1} \end{pmatrix}.$$

The first representation of the theorem is effectively the range of $H\tilde{H}$, or in matrix terms, $\mathcal{A}\tilde{\mathcal{A}}$. Since $A_k = 0$ for $k > N$, the range of $\mathcal{A}$, and therefore also that of $\mathcal{A}\tilde{\mathcal{A}}$, lies in the space of sequences $(u_k)_{k=0}^{\infty}$ for which $u_k = 0$ for $k \geq N$. As $\mathcal{A}\tilde{\mathcal{A}}$ is a projection, a representation of the multiplicity is obtained by restricting $\mathcal{A}\tilde{\mathcal{A}}$ to the subspace of such sequences, and by truncating the sequences to length N we get the matrix product BC, also a projection.

We have, by Proposition 1.3(ii) and Theorem 3.1, that $\tilde{\mathcal{A}}\mathcal{A}\tilde{\mathcal{A}} = \tilde{\mathcal{A}}$, and since the range of $\mathcal{A}\tilde{\mathcal{A}}$ lies in the space of sequences $(u_k)_{k=0}^{\infty}$ for which $u_k = 0$ for $k \geq N$ we truncate the sequences in the domain to length N. We then obtain $CBC = C$. But now it follows that CB is a projection matrix and its range is isomorphic to the range of BC. Finally, since $CBC = C$, we conclude that the matrices CB and C have the same range. □

Note that the second representation is equivalent to a space of poles of order less than or equal to N, but it is not the range of $\tilde{\mathcal{A}}$; it is obtained from the latter by truncating the principal parts to order N.

We observe that the projection matrices of Theorem 6.1 provide us with a generalization of the Riesz projection. Indeed, suppose that our polynomial is of the form $A(z) = A_0 + (z - z_0)I$. Then, Theorem 6.1 says that, if z_0 is an isolated point of the spectrum, $\tilde{A}_{-1}$ is a projection whose range is a representation of the multiplicity.

We conclude the section with two further sufficient conditions for the multiplicity to be finite.

Proposition 6.2. *Let A be a polynomial of degree N and let z_0 be an isolated point of the spectrum. If $A_1, \dots, A_N$ are compact or $\tilde{A}_{-1}, \dots, \tilde{A}_{-N}$ are compact then the multiplicity of A at z_0 is finite.*

Proof. Suppose $A_1, \dots, A_N$ are compact. Then, by Theorem 6.1,

$$\begin{pmatrix} A_1 & \cdots & A_N \\ \vdots & \iddots & \\ A_N & & \end{pmatrix} \begin{pmatrix} \tilde{A}_{-1} & \cdots & \tilde{A}_{-N} \\ \vdots & \ddots & \vdots \\ \tilde{A}_{-N} & \cdots & \tilde{A}_{-2N+1} \end{pmatrix}$$

is a compact projection, hence of finite rank.

Now suppose that $\tilde{A}_{-1}, \dots, \tilde{A}_{-N}$ are compact. The proof will be finished as soon as we show that $\tilde{A}_{-k}$ is compact for every $k \geq 1$. By Proposition 1.3 and Theorem 3.1, $\operatorname{ran} S$ is contained in the space of polynomials of degree less than N. Now we show that an equivalent norm for $\operatorname{ran} S$, as a representative of the multiplicity, is

$$\left\| \sum_{j=0}^{N-1} (z - z_0)^j u_j \right\|_{\operatorname{ran} S} := \sum_{j=0}^{N-1} \|u_j\|.$$

Indeed, it was shown in Section 1 that a norm for $\operatorname{ran} S$ is given by $p(f) = \sup_{C_r} \|f\|$ where $0 < r \leq 1$ and r is sufficiently small, C_r being the circumference of radius r centred at z_0. Then, clearly, $p \leq \|\cdot\|_{\operatorname{ran} S}$. Conversely, suppose that a sequence f_n of elements in $\operatorname{ran} S$ converges to 0 in the norm p. Then, by well-known results in analytic-function theory, for any $j \geq 0$, the sequence of the derivatives $\{f_n^{j)}(z_0)\}_{n\in\mathbb{N}}$ converges to zero. This proves that the sequence $\|f_n\|_{\operatorname{ran} S}$ converges to zero and so the norms are equivalent.

Now we show that $\tilde{A}_{-k}$ is compact for every $k \geq 1$. Consider the operator

$$T : \mathcal{O}(\Omega, E) \to E\,, \qquad Tf = \frac{1}{2\pi i}\int_{\partial\Omega} A(\zeta)^{-1} f(\zeta)\, d\zeta,$$

where Ω is a small disc of radius greater than r centred at z_0. We have that $T = TS$, as a consequence of Proposition 1.2 and the fact that $T(Af) = 0$ for every $f \in \mathcal{O}(\Omega, E)$. Now, for any $k \geq 0$ and any $u \in E$, the function $f(z) = (z - z_0)^k u$ satisfies $Tf = \tilde{A}_{-k-1}u$. We now show that the restriction $T : \operatorname{ran} S \to E$ is compact. Indeed, if f_n is a sequence in $\operatorname{ran} S$

$$f_n(z) = \sum_{j=0}^{N-1} (z - z_0)^j f_{j,n}$$

such that $\|f_{j,n}\|$ is bounded with respect to $j \in \{0, \ldots, N-1\}$, $n \in \mathbb{N}$, then,

$$Tf_n = \sum_{j=0}^{N-1} \tilde{A}_{-j-1} f_{j,n}$$

admits a convergent subsequence since $\tilde{A}_{-1}, \ldots, \tilde{A}_{-N}$ are compact. Now, if v_n is a bounded sequence in E, then for any $k \geq 1$,

$$\tilde{A}_{-k}v_n = T((z - z_0)^{k-1}v_n) = TS((z - z_0)^{k-1}v_n)$$

admits a convergent subsequence, since $T|_{\operatorname{ran} S}$ is compact. This proves that $\tilde{A}_{-k}$ is compact for every $k \geq 1$ and concludes the proof. □

7. An earlier definition of multiplicity as a quotient

In a series of works, B. F. Wyman and M. K. Sain developed a notion of "zero-module", see [3] and the reference therein. This zero-module resembles the definition of multiplicity given in [1] and summarized in Section 1. Let us recall the construction in [3]. Let G be a rational $m \times n$ function. It can be regarded as a $\mathbb{C}(z)$-linear map from $\mathbb{C}^n(z)$ to $\mathbb{C}^m(z)$, where $\mathbb{C}(z)$ is the field of rational scalar functions, and, for any integer p, $\mathbb{C}^p(z)$ is the set of vectors of dimension p whose components are in $\mathbb{C}(z)$. Analogously, $\mathbb{C}[z]$ is the ring of complex polynomials, and $\mathbb{C}^p[z]$ stands for the set of vectors of dimension p whose components are in $\mathbb{C}[z]$. The zero-module is the $\mathbb{C}[z]$-module

$$Z(G) := \frac{G^{-1}(\mathbb{C}^m(z)) + \mathbb{C}^n[z]}{\ker G + \mathbb{C}^n[z]}.$$

They proved that $Z(G)$ is a finite-dimensional $\mathbb{C}(z)$-vector space, and constructed a basis for it by using the Smith-McMillan canonical form, which is known to be related to Jordan chains. They saw that $Z(G)$ measures a sort of "global multiplicity" in $\mathbb{C}$. This definition can be easily adapted to define a "zero-module" measuring a multiplicity in a subset $\Omega \subset \mathbb{C}$. Nonetheless, it is by no means clear how to extend this construction to the infinite-dimensional case, let alone the non-Fredholm case, which is the main point of [1]. One of the difficulties to overcome would be to prove that the quotient involved in the definition is a Banach space.

Acknowledgment

The second author has been supported by the Spanish Ministry of Science and Technology under Grants BFM2000-0797 and REN2003-00707, and by the Spanish MECD under Grant AP2000-3316.

References

[1] J. Arason and R. Magnus, *An algebraic multiplicity theory for analytic operator-valued functions.* Math. Scand. **82** (1998), 265–286.

[2] J. A. Ball, I. Gohberg and L. Rodman, *Interpolation of Rational Matrix Functions*, Operator Theory: Advances and Applications 45. Birkhäuser, Basel 1990.

[3] J. A. Ball, M. Rakowski and B. F. Wyman, *Coupling operators, Wedderburn-Forney spaces, and generalized inverses,* Linear Algebra and Applications **203/204** (1994), 111–138.

[4] I. Gohberg, S. Goldberg, and M. A. Kaashoek: *Classes of Linear Operators, vol. 1.*, Operator Theory: Advances and Applications 49, Birkhäuser, Bassel 1990.

[5] I. C. Gohberg, M. A. Kaashoek and D. C. Lay, *Equivalence, Linearization, and Decomposition of Holomorphic Operator Functions.* J. Funct. Anal. **28** (1978), 102–144.

[6] I. Gohberg and L. Rodman, *Analytic operator-valued functions with prescribed local data,* Acta Math. (Szeged) **45** (1983), 189–299.

[7] I. C. Gohberg and E. I. Sigal, *An Operator Generalization of the Logarithmic Residue Theorem and the Theorem of Rouché.* Math. Sbornik **84(126)** (1971), 607–629. English Trans.: Math. USSR Sbornik **13** (1971), 603–625.

[8] J. López-Gómez and C. Mora-Corral, *Characterizing the Existence of Local Smith Forms for $\mathcal{C}^\infty$ Families of Matrix Operators,* in *Trends in Banach Spaces and Operator Theory*, A. Kamińska (ed.). Contemporary Mathematics **321**, 139–151. (AMS, Providence, 2003).

[9] A. S. Markus and E. I. Sigal, *The multiplicity of the characteristic number of an analytic operator function.* (Russian) Mat. Issled. **5** (1970), 129–147.

[10] V. V. Peller, *Hankel operators and their applications.* Springer Monographs in Mathematics. Springer, New York, 2003.

[11] C. R. Putnam, *Commutation properties of Hilbert space operators and related topics.* Ergebnisse der Mathematik und ihrer Grenzgebiete 36. Springer, Berlin 1967.

Robert Magnus
Science Institute. University of Iceland. Dunhagi 3.
IS-107 Reykjavik. Iceland
E-mail: robmag@raunvis.hi.is

Carlos Mora-Corral
Facultad de Matemáticas. Departamento de Matemática Aplicada.
Universidad Complutense de Madrid.
28040. Madrid. Spain
E-mail: Carlos_Mora@mat.ucm.es