

The implicit function theorem and multibump solutions of periodic partial differential equations

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Abstract

A modification of the implicit function theorem is advanced for cases where the continuity of the derivative fails. It is applied to a superposition principle for periodic partial differential equations. The assumption of the principle, that there should exist a non-degenerate solution, is studied and instances of it realized using perturbation arguments and scaling. The positivity of solutions is considered.

1 The modified implicit function theorem

The implicit function theorem provides the justification for many perturbation arguments, where solutions to an equation are sought for “small ϵ ” near to a known solution for $\epsilon = 0$. There are, however, cases where the theorem is not immediately applicable because of a failure of continuity of the derivative in a peculiarly infinite-dimensional fashion. A version of the implicit function theorem, in a form commonly used in perturbation problems, and with fairly minimal hypotheses, is as follows. Note that \mathbf{R}_+ denotes the non-negative real numbers.

Theorem 1.1 *Let E and F be real Banach spaces, and let $f : \mathbf{R}_+ \times E \rightarrow F$. Assume that*

- (1) $f(\epsilon, \cdot)$ is C^1 for each $\epsilon \geq 0$;
- (2) there exists $x_0 \in E$ such that $f(0, x_0) = 0$;
- (3) $D_x f(0, x_0)$ is invertible;
- (4) $\lim_{\epsilon \rightarrow 0^+} f(\epsilon, x_0) = 0$;
- (5) $\lim_{\epsilon \rightarrow 0^+, x \rightarrow x_0} D_x f(\epsilon, x) = D_x f(0, x_0)$ in the operator norm topology.

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Then there exist $\epsilon_0 > 0$ and a neighbourhood U of x_0 in E such that for each ϵ in the range $0 < \epsilon < \epsilon_0$ there exists a unique solution $x = x_\epsilon$ of $f(\epsilon, x) = 0$ in U . Moreover $\lim_{\epsilon \rightarrow 0} x_\epsilon = x_0$.

If, furthermore, $f(\epsilon, x)$ is a continuous function of ϵ for $\epsilon > 0$ and for all x in a neighbourhood of x_0 , then the solution x_ϵ depends continuously on ϵ .

A bounded linear operator from E to F is said here to be invertible (cf. condition (3)) if it has an inverse that is a bounded linear operator from F to E . We say that x_0 is a *non-degenerate solution* of the *limit equation* $f(0, x) = 0$ when conditions (2) and (3) hold.

Essentially the same theorem holds (though without its final conclusion) if ϵ is a discrete parameter or, more generally, if we have a family of problems $f_\nu(x) = 0$ indexed by ν which runs through a set N . A simple way to handle limits is to assume that there exists a real-valued function $r : N \rightarrow \mathbf{R}_+$ (the non-negative reals), unbounded on N and take as our limit equation the problem $\lim_{r(\nu) \rightarrow \infty} f_\nu(x) = 0$ if the limit exists. The translation of conditions (1–5) of Theorem 1.1 to this case should be obvious.

We shall consider problems in which the limit of condition (5) does not hold in the operator norm topology, in which case the perturbed solution may not exist even if the limit holds in the strong operator topology. Therefore we need to impose some further conditions to guarantee the same conclusions as in Theorem 1.1.

Theorem 1.2 *Let E and F be real Banach spaces, and let $f : \mathbf{R}_+ \times E \rightarrow F$. Assume that*

- (1) $f(\epsilon, \cdot)$ is C^1 for each $\epsilon \geq 0$;
- (2) there exists $x_0 \in E$ such that $f(0, x_0) = 0$;
- (3) $D_x f(0, x_0)$ is invertible;
- (4) $\lim_{\epsilon \rightarrow 0^+} f(\epsilon, x_0) = 0$;
- (5) for all sufficiently small $\epsilon > 0$ the operator $D_x f(\epsilon, x_0)$ is invertible and $\|D_x f(\epsilon, x_0)^{-1}\|$ is uniformly bounded as $\epsilon \rightarrow 0^+$;
- (6) $\lim_{\epsilon \rightarrow 0^+, x \rightarrow x_0} \|D_x f(\epsilon, x) - D_x f(\epsilon, x_0)\| = 0$.

Then there exist $\epsilon_0 > 0$ and a neighbourhood U of x_0 in E such that for each ϵ in the range $0 < \epsilon < \epsilon_0$ there exists a unique solution $x = x_\epsilon$ of $f(\epsilon, x) = 0$ in U . Moreover $\lim_{\epsilon \rightarrow 0} x_\epsilon = x_0$.

If, furthermore, $f(\epsilon, x)$ is a continuous function of ϵ for $\epsilon > 0$ and for all x in a neighbourhood of x_0 , and the map $\epsilon \mapsto D_x f(\epsilon, x_0)$ is continuous in the strong operator topology, then the solution x_ϵ depends continuously on ϵ .

Proof Let

$$g(\epsilon, x) = D_x f(\epsilon, x_0)^{-1} f(\epsilon, x)$$

where $0 < \epsilon < \epsilon_1$ and $D_x f(\epsilon, x_0)$ is invertible for $0 < \epsilon < \epsilon_1$. We apply Theorem 1.1 to $g(\epsilon, x)$. We have

$$\lim_{\epsilon \rightarrow 0^+} g(\epsilon, x_0) = 0$$

by virtue of conditions (4) and (5). We have

$$\begin{aligned} D_x g(\epsilon, x) - D_x g(0, x_0) &= D_x f(\epsilon, x_0)^{-1} D_x f(\epsilon, x) - I \\ &= D_x f(\epsilon, x_0)^{-1} (D_x f(\epsilon, x) - D_x f(\epsilon, x_0)) \end{aligned}$$

and this tends to 0 in the operator norm topology as $\epsilon \rightarrow 0^+$, $x \rightarrow x_0$ by virtue of conditions (5) and (6).

Suppose now that $D_x f(\epsilon, x_0)$ is a continuous function of ϵ in the strong operator topology. Then the same is true of $D_x f(\epsilon, x_0)^{-1}$. Indeed for each vector $v \in F$ we have

$$\begin{aligned} &\left(D_x f(\epsilon', x_0)^{-1} - D_x f(\epsilon, x_0)^{-1} \right) v \\ &= D_x f(\epsilon', x_0)^{-1} (D_x f(\epsilon, x_0) - D_x f(\epsilon', x_0)) D_x f(\epsilon, x_0)^{-1} v \rightarrow 0 \end{aligned}$$

as $\epsilon' \rightarrow \epsilon$ because of condition (5) and the strong continuity of $D_x f(\cdot, x_0)$. Hence if, in addition, $f(\epsilon, x)$ is a continuous function of ϵ we can conclude the same for $g(\epsilon, x)$, and so, by Theorem 1.1, the solution x_ϵ depends continuously on ϵ .

The real difficulty in applying Theorem 1.2 lies in verifying condition (5). The further condition (6) is not onerous. It is for example satisfied if the second derivative $D_x^2 f(\epsilon, x)$ has a uniform bound in norm as $x \rightarrow x_0$ and $\epsilon \rightarrow 0$, quite a natural hypothesis, though not one we shall make use of.

A Fredholm operator of index 0 has the property that it is surjective if and only if it is injective; so the verification of (5) is facilitated if $D_x f(\epsilon, x_0)$ is such an operator. In practice we shall verify (5) by applying the following proposition, the proof of which is straightforward.

Lemma 1.3 *Let $\{T_\epsilon\}_{0 < \epsilon < \epsilon_2}$, be a family of Fredholm operators of index 0 from E to F . Suppose that there do not exist sequences $\epsilon_\nu \rightarrow 0$, $x_\nu \in E$ such that $\|x_\nu\| = 1$ and $T_{\epsilon_\nu} x_\nu \rightarrow 0$. Then there exists $\epsilon_1 > 0$ such that for all ϵ in the range $0 < \epsilon < \epsilon_1$ the operator T_ϵ is invertible and there exists a constant K independent of ϵ such that $\|T_\epsilon^{-1}\| < K$.*

More generally let $\{T_\nu\}_{\nu \in N}$ be a family of Fredholm operators of index 0 indexed by a set N , and let $r : N \rightarrow \mathbf{R}_+$ be an unbounded function. A subset $N' \subset N$ is said to be admissible if r is unbounded on N' . Suppose that there exists no admissible set N' such that vectors x_ν can be chosen for each $\nu \in N'$ satisfying $\|x_\nu\| = 1$, $\lim_{r(\nu) \rightarrow \infty, \nu \in N'} T_\nu x_\nu = 0$. Then there exists $K > 0$ such that $\|T_\nu^{-1}\|$ has a uniform bound for $\nu \in N$, $r(\nu) \geq K$.

Ignoring the difference between Theorems 1.1 and 1.2 can lead to incorrect conclusions. A simple example will illustrate this. Let E be the Hilbert space l^2 of all real square summable sequences $x = \{x_i\}_{i=1}^{\infty}$. For each positive integer n we define $f_n : E \rightarrow E$ by

$$f_n(x)_i = \begin{cases} x_i + \frac{1}{in} & \text{if } i \neq n \\ \frac{1}{n^2} & \text{if } i = n \end{cases}$$

Note that there is no solution to $f_n(x) = 0$. However we have

$$f_n(x) = x - \langle x, e_n \rangle e_n + \frac{1}{n} a$$

where $a = \{\frac{1}{i}\}$ and e_n is the n th basis vector in the natural orthonormal basis. It seems that as $n \rightarrow \infty$ we obtain the limit equation $x = 0$. However condition (5) of Theorem 1.1 (reformulated for a discrete parameter) does not hold; the operator P_n given by $P_n x = \langle x, e_n \rangle e_n$ converges to 0 in the strong operator topology but not in the operator norm topology.

The properties of $f(\epsilon, x)$ for $\epsilon = 0$ in Theorem 1.2 are essentially irrelevant. In fact $f(\epsilon, x)$ does not have to be defined for $\epsilon = 0$, which means that we may drop conditions (2) and (3) completely and only assert condition (1) for $\epsilon > 0$. In particular x_0 is then not posited as a solution of an equation. To derive this version of the theorem from Theorem 1.2 it suffices to define $f(0, x) = T(x - x_0)$ where T is an arbitrarily chosen isomorphism from E onto F .

The rationale behind our method, with its unnecessary hypotheses, is that the element x_0 usually arises in practice as a solution of a limit equation. A family of problems $f_\nu(x) = 0$ may have a limit problem $\lim f_\nu(x) = 0$ with a non-degenerate solution. It then appears natural to apply the Implicit Function Theorem. This requires us to check that the derivative converges in the operator norm; if it does we use Theorem 1.1. If it doesn't we apply Theorem 1.2, in the course of which the verification of condition (5) may require further hypotheses. An early attempt to formulate this method in a concrete example was described in [10].

The contents of the paper are as follows. In section 2 we prove, as an application of Theorem 1.2, a superposition property for the solutions of periodic partial differential equations. In sections 3, 4 and 5 we study the conditions imposed on the equation of section 2. In particular, in sections 4 and 5 we exhibit a large class of equations possessing the non-degenerate solutions required in section 2. In section 6 we show that the solutions of section 2 are positive under appropriate conditions.

2 Superposition of solutions of periodic PDEs

Consider the problem

$$Lu + F(x, u, \nabla u) = 0 \tag{2.1}$$

over all \mathbf{R}^n , where

(a) L is a linear, second order differential operator in \mathbf{R}^n of the form

$$L = \sum_{i,j} b_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$$

with coefficients b_{ij} that are bounded, measurable functions, periodic with common period lattice Λ ;

(b) F is a C^2 function of u , ∇u , and a measurable function of x , periodic in x with period lattice Λ .

For convenience in writing partial derivatives of F we denote the $n + 1$ variables of F (those after x) by $\omega_0, \omega_1, \dots, \omega_n$. For each function u defined in \mathbf{R}^n we define the vector function

$$\kappa u = (\kappa_0 u, \kappa_1 u, \dots, \kappa_n u) = (u, \nabla u).$$

In the interests of readability we shall abuse notation for functions by referring, for example, to the L^2 norm of $F(x, u, \nabla u)$, where what is meant is the L^2 norm of the function properly denoted by $x \mapsto F(x, u(x), \nabla u(x))$, or by $F(\cdot, u, \nabla u)$ or by $\mathbf{F}(u)$ where \mathbf{F} is the operator induced by F . For the same reason a variety of notations will be used for partial derivatives; for example $D_i g$ in some places, and in others $\frac{\partial F}{\partial \omega_i}$ and even a mix of the two in the same formula. This will reflect different roles played by different variables.

We make two further assumptions about F . Note that $\mathcal{L}(X, Y)$ denotes the space of bounded linear operators from X to Y , and $\mathcal{L}_2(X \times X, Y)$ the space of bounded bilinear operators from $X \times X$ to Y . We use H^2 to denote $W^{2,2}(\mathbf{R}^n)$ and L^2 to denote $L^2(\mathbf{R}^n)$. Their norms are denoted by $\|u\|_{2,2}$ and $\|u\|_2$ respectively.

(c) F , $\frac{\partial F}{\partial \omega_i}$ and $\frac{\partial^2 F}{\partial \omega_i \partial \omega_j}$ define maps

$$\mathbf{F} : H^2 \rightarrow L^2, \quad \mathbf{F}_i : H^2 \rightarrow \mathcal{L}(H^2, L^2), \quad \mathbf{F}_{ij} : H^2 \rightarrow \mathcal{L}_2(H^2 \times H^2, L^2)$$

given by

$$\begin{aligned} \mathbf{F}(u) &= F(\cdot, \kappa u), \\ \mathbf{F}_i(u)v &= \frac{\partial F}{\partial \omega_i}(\cdot, \kappa u) \kappa_i v, \\ \mathbf{F}_{ij}(u)(v, w) &= \frac{\partial^2 F}{\partial \omega_i \partial \omega_j}(\cdot, \kappa u) (\kappa_i v) (\kappa_j w). \end{aligned}$$

Moreover \mathbf{F} , \mathbf{F}_i and \mathbf{F}_{ij} map bounded sets to bounded sets.

(d) Let $u, v \in H^2$ and let $(w_\nu)_{\nu=1}^\infty$ be a sequence in H^2 with limit 0 in the weak topology. Then for each i and j the sequence

$$\left(\mathbf{F}_{ij}(u)(\kappa_i v)(\kappa_j w_\nu) \right)_{\nu=1}^\infty$$

has the limit 0 in the L^2 norm, and the limit is attained uniformly with respect to u in bounded subsets of H^2 .

Later we shall describe growth conditions on F that imply (c) and (d). It even seems plausible that the existence part of (c) implies the boundedness part, and even (d), in view of surprising properties of Nemytskii operators (see [8]). Note that (d) implies that the operator

$$w \mapsto \frac{\partial^2 F}{\partial \omega_i \partial \omega_j} \left(x, \kappa u \right) (\kappa_i v)(\kappa_j w)$$

is compact from H^2 to L^2 , but we are saying a little more in demanding uniformity of convergence w.r.t. u .

Lemma 2.1 *Under condition (c) the non-linear operator \mathbf{F} is continuously differentiable and its derivative is given by*

$$D\mathbf{F}(u)v = \sum_{i=0}^n \mathbf{F}_i(u)\kappa_i v$$

Proof We have that

$$\begin{aligned} \mathbf{F}(u+v) - \mathbf{F}(u) - \sum_{i=0}^n \mathbf{F}_i(u)\kappa_i v &= \sum_{i=0}^n \int_0^1 \left(\frac{\partial F}{\partial \omega_i} \left(x, \kappa u + t\kappa v \right) - \frac{\partial F}{\partial \omega_i} \left(x, \kappa u \right) \right) \kappa_i v dt \\ &= \sum_{i=0}^n \sum_{j=0}^n \int_0^1 \int_0^1 \frac{\partial^2 F}{\partial \omega_i \partial \omega_j} \left(x, \kappa u + s t \kappa v \right) \kappa_i v \kappa_j v t ds dt. \end{aligned}$$

By condition (c) we have

$$\left\| \mathbf{F}(u+v) - \mathbf{F}(u) - \sum_{i=0}^n \mathbf{F}_i(u)\kappa_i v \right\|_2 \leq K \|v\|_{2,2}^2$$

where K can be chosen uniformly for v in bounded sets. This implies the differentiability of \mathbf{F} and the formula for the derivative. That the derivative is continuous follows from

$$\left(\mathbf{F}_i(u+v) - \mathbf{F}_i(u) \right) w = \sum_{j=0}^n \int_0^1 \frac{\partial^2 F}{\partial \omega_i \partial \omega_j} \left(x, \kappa u + t\kappa v \right) \kappa_i w \kappa_j v dt$$

and the boundedness property of condition (c).

By a *non-degenerate solution* of (2.1) we mean a solution $u \in H^2$ such that the linear operator

$$v \mapsto Lv + \sum_{i=0}^n \frac{\partial F}{\partial \omega_i}(x, \kappa u) \kappa_i v$$

from H^2 to L^2 is invertible.

We shall be concerned with families indexed by elements of Λ^m . For each $\xi = (\xi_1, \dots, \xi_m) \in \Lambda^m$ we let

$$\mu(\xi) = \min_{i \neq j} \|\xi_i - \xi_j\|,$$

using the Euclidean norm in Λ . We define $\delta_{ij}(\xi)$, or simply δ_{ij} when ξ is understood, to be $\xi_i - \xi_j$. This is a vector in \mathbf{R}^n and it satisfies $\delta_{ij} + \delta_{jk} = \delta_{ik}$, $\delta_{ii} = 0$. Limits of families indexed by Λ^m are understood in the following sense. Let x_ξ , $\xi \in \Lambda^m$, be such a family in a normed space. Then we say that $\lim x_\xi = y$ if for each $\epsilon > 0$ there exists $M > 0$ such that $\|x_\xi - y\| < \epsilon$ whenever $\mu(\xi) \geq M$. The role of subsequence will be taken by the restriction of the family to a subset $X \subset \Lambda^m$ on which $\mu(\xi)$ is unbounded. We shall refer to such a subset as an *admissible* subset (cf. section 1).

We now come to our main application of the Modified Implicit Function Theorem.

Theorem 2.2 *Assume that L and F satisfy the conditions (a), (b), (c) and (d). Let ϕ_k , $k = 1, \dots, m$, be non-degenerate solutions (not necessarily distinct) of $Lu + F(x, u, \nabla u) = 0$ in H^2 . Then there exist a neighbourhood U of 0 in H^2 and $K > 0$ such that for all $\xi \in \Lambda^m$ such that $\mu(\xi) > K$ there exists a unique solution u of $Lu + F(x, u, \nabla u) = 0$ of the form*

$$u(x) = \sum_{k=1}^m v_k(x + \xi_k)$$

where $v_k \in U + \phi_k$. Moreover $v_k \rightarrow \phi_k$ in H^2 as $\mu(\xi) \rightarrow \infty$.

Note. For the most part only variational problems of this type have been treated, see e.g. [6], though not always by variational methods. Our methods can be compared to those of Angenent [2] who viewed the problem as a generalization of the shadowing lemma of dynamical systems theory, and used the same non-degeneracy condition on the solutions ϕ_k , and a somewhat similar approach. In [6] the solutions ϕ_k are assumed to be critical points with the lowest positive critical value of the corresponding functional. The same is true of a recent paper [1] in which the non-linearity is allowed to be non-local.

Proof We seek a solution of the form

$$u(x) = \sum_{k=1}^{m-1} \phi_k(x + \xi_k) + v(x + \xi_m).$$

where the unknown function v is near ϕ_m . Then v satisfies the problem

$$\Gamma(\xi, v) = 0$$

where

$$\Gamma(\xi, v) = Lv + F\left(x, \sum_{k=1}^{m-1} \kappa \phi_k(x + \delta_{km}) + \kappa v\right) - \sum_{k=1}^{m-1} F\left(x, \kappa \phi_k(x + \delta_{km})\right)$$

and, we recall, $\delta_{ij} = \xi_i - \xi_j$ and $\kappa u = (u, \nabla u)$. By Lemma 2.1 the map $v \mapsto \Gamma(\xi, v)$, from H^2 to L^2 , is continuously differentiable for each ξ .

We shall show that

$$\lim_{\mu(\xi) \rightarrow \infty} \Gamma(\xi, v) = Lv + F(x, \kappa v)$$

so that we have a limit problem $Lv + F(x, \kappa v) = 0$ with the non-degenerate solution ϕ_m . First we write

$$\begin{aligned} & F\left(x, \sum_{k=1}^{m-1} \kappa \phi_k(x + \delta_{km}) + \kappa v\right) - \sum_{k=1}^{m-1} F\left(x, \kappa \phi_k(x + \delta_{km})\right) - F(x, \kappa v) \\ &= \sum_{j=1}^{m-1} \left[F\left(x, \sum_{k=j}^{m-1} \kappa \phi_k(x + \delta_{km}) + \kappa v\right) - F\left(x, \sum_{k=j+1}^{m-1} \kappa \phi_k(x + \delta_{km}) + \kappa v\right) - F\left(x, \kappa \phi_j(x + \delta_{jm})\right) \right]. \end{aligned} \tag{2.2}$$

Now we use a result from elementary calculus. If $g(y)$ is a C^2 function of $y \in \mathbb{R}^{n+1}$ then

$$g(u+v) - g(u) - g(v) = \sum_{p,q} \int_S \frac{\partial^2 g}{\partial y_p \partial y_q}(su + tv) u_p v_q ds dt$$

where S is the unit square $0 \leq s, t \leq 1$. Applying this to one term on the right-hand side of (2.2) and dropping the summation over p, q we obtain the expression

$$\begin{aligned} & \int_S \frac{\partial^2 F}{\partial \omega_p \partial \omega_q} \left(x, s \kappa \phi_j(x + \delta_{jm}) + t \left(\sum_{k=j+1}^{m-1} \kappa \phi_k(x + \delta_{km}) + \kappa v \right) \right) \\ & \quad \cdot \kappa_p \phi_j(x + \delta_{jm}) \left(\sum_{k=j+1}^{m-1} \kappa_q \phi_k(x + \delta_{km}) + \kappa_q v \right) ds dt. \end{aligned}$$

Recall that we wish to show that this tends to 0 in L^2 as $\mu(\xi) \rightarrow \infty$. This will follow if we show that the integrand tends to 0 in L^2 uniformly w.r.t. s, t in S . This follows from condition (d) as we now show. The integrand expands into two types of term: the first is

$$\frac{\partial^2 F}{\partial \omega_p \partial \omega_q} \left(x, s \kappa \phi_j(x + \delta_{jm}) + t \left(\sum_{k=j+1}^{m-1} \kappa \phi_k(x + \delta_{km}) + \kappa v \right) \right) \kappa_p \phi_j(x + \delta_{jm}) \kappa_q v$$

Since $\phi_j(x + \delta_{jm}) \rightarrow 0$ weakly in H^2 as $\mu(\xi) \rightarrow \infty$ (note here that $j \neq m$) this tends to 0 in L^2 uniformly w.r.t. s, t in S by condition (d). The second kind of term is

$$\frac{\partial^2 F}{\partial \omega_p \partial \omega_q} \left(x, s\kappa\phi_j(x + \delta_{jm}) + t \left(\sum_{k=j+1}^{m-1} \kappa\phi_k(x + \delta_{km}) + \kappa v \right) \right) \kappa_p \phi_j(x + \delta_{jm}) \kappa_q \phi_l(x + \delta_{lm})$$

where $j+1 \leq l \leq m-1$. Since we want to prove that this has the limit 0 in the L^2 norm we may first replace x by $x - \delta_{jm}$. This gives

$$\frac{\partial^2 F}{\partial \omega_p \partial \omega_q} \left(x, s\kappa\phi_j + t \left(\sum_{k=j+1}^{m-1} \kappa\phi_k(x + \delta_{kj}) + \kappa v(x - \delta_{jm}) \right) \right) \kappa_p \phi_j \kappa_q \phi_l(x + \delta_{lj})$$

This tends to 0 in the L^2 norm by condition (d) (and because $l \neq j$).

Having shown that a limit problem exists and has the non-degenerate solution $v = \phi_m$ we would like to apply the implicit function theorem. However it is clear that in general the derivative of $\Gamma(\xi, v)$ with respect to v does not converge in the operator norm topology as $\mu(\xi) \rightarrow \infty$; it converges only in the strong operator topology. We therefore apply Theorem 1.2, in a form adapted to a family of functions indexed by $\xi \in \Lambda^n$. The lengthiest part of the argument is the verification of condition (5).

The derivative of Γ at the solution ϕ_m is the linear map $T_\xi : H^2 \rightarrow L^2$ given by

$$T_\xi w = D_v \Gamma(\xi, \phi_m) w = Lw + \sum_{p=0}^n \frac{\partial F}{\partial \omega_p} \left(x, \sum_{k=1}^m \kappa\phi_k(x + \delta_{km}) \right) \kappa_p w.$$

We first show that this is a Fredholm operator of index 0. We write

$$\begin{aligned} T_\xi w &= Lw + \sum_{p=0}^n \frac{\partial F}{\partial \omega_p} \left(x, \kappa\phi_m \right) \kappa_p w \\ &\quad + \sum_{p=0}^n \left[\frac{\partial F}{\partial \omega_p} \left(x, \sum_{k=1}^m \kappa\phi_k(x + \delta_{km}) \right) - \frac{\partial F}{\partial \omega_p} \left(x, \kappa\phi_m \right) \right] \kappa_p w. \end{aligned}$$

The first two terms constitute together an invertible operator (by the non-degeneracy assumption). The third term is a compact operator; in fact we can write

$$\begin{aligned} &\left[\frac{\partial F}{\partial \omega_p} \left(x, \sum_{k=1}^m \kappa\phi_k(x + \delta_{km}) \right) - \frac{\partial F}{\partial \omega_p} \left(x, \kappa\phi_m \right) \right] \kappa_p w \\ &= \sum_{q=0}^n \int_0^1 \frac{\partial^2 F}{\partial \omega_p \partial \omega_q} \left(x, t \sum_{k=1}^{m-1} \kappa\phi_k(x + \delta_{km}) + \kappa\phi_m \right) \sum_{k=1}^{m-1} \kappa_q \phi_k(x + \delta_{km}) \kappa_p w dt \end{aligned}$$

and the compactness follows from condition (d).

We may therefore verify condition (5) of Theorem 1.2 by showing that a contradiction ensues from the assumption that there exists a family $w_\xi \in H^2$ with $\|w_\xi\|_{2,2} = 1$, indexed by ξ in an admissible set $X \subset \Lambda^m$ (that is, in a set X on which $\mu(\xi)$ is unbounded), such that $T_\xi w_\xi \rightarrow 0$ in L^2 as $\mu(\xi) \rightarrow \infty$.

Having made this assumption we choose an admissible subset $X' \subset X$ such that the limits

$$\lim_{\mu(\xi) \rightarrow \infty, \xi \in X'} w_\xi(x + \delta_{mj}) = y_j$$

exist in the weak topology on H^2 for $j = 1, \dots, m$. By assumption $T_\xi w_\xi \rightarrow 0$ in L^2 , so that, replacing x by $x + \delta_{mj}$, we find that

$$Lw_\xi(x + \delta_{mj}) + \sum_{p=0}^n \frac{\partial F}{\partial \omega_p} \left(x, \sum_{k=1}^m \kappa \phi_k(x + \delta_{kj}) \right) \kappa_p w_\xi(x + \delta_{mj})$$

tends to 0 in L^2 . But we claim that this converges as a distribution to

$$Ly_j + \sum_{p=0}^n \frac{\partial F}{\partial \omega_p} \left(x, \kappa \phi_j \right) \kappa_p y_j$$

so that, by non-degeneracy, $y_j = 0$ for each j . To prove the claim about the distribution limit we first observe that

$$Lw_\xi(x + \delta_{mj}) \rightarrow Ly_j$$

and

$$\frac{\partial F}{\partial \omega_p} \left(x, \kappa \phi_j \right) \kappa_p w_\xi(x + \delta_{mj}) \rightarrow \frac{\partial F}{\partial \omega_p} \left(x, \kappa \phi_j \right) \kappa_p y_j$$

weakly in L^2 . It therefore remains to show that

$$\left(\frac{\partial F}{\partial \omega_p} \left(x, \sum_{k=1}^m \kappa \phi_k(x + \delta_{kj}) \right) - \frac{\partial F}{\partial \omega_p} \left(x, \kappa \phi_j \right) \right) \kappa_p w_\xi(x + \delta_{mj})$$

tends to 0 as a distribution. Let χ be a test function for distributions. We must show that

$$\int_{\mathbf{R}^n} \left(\frac{\partial F}{\partial \omega_p} \left(x, \sum_{k=1}^m \kappa \phi_k(x + \delta_{kj}) \right) - \frac{\partial F}{\partial \omega_p} \left(x, \kappa \phi_j \right) \right) \kappa_p w_\xi(x + \delta_{mj}) \chi dx$$

tends to 0. We can write this as

$$\sum_q \int_{\mathbf{R}^n} \int_0^1 \frac{\partial^2 F}{\partial \omega_p \partial \omega_q} \left(x, t \sum_{\substack{k=1 \\ k \neq j}}^m \kappa \phi_k(x + \delta_{kj}) + \kappa \phi_j \right) \cdot \left(\sum_{\substack{k=1 \\ k \neq j}}^m \kappa_q \phi_k(x + \delta_{kj}) \right) \kappa_p w_\xi(x + \delta_{mj}) \chi \, dt \, dx.$$

Dropping the summation over q we bound this by the product of L^2 norms

$$\left\| \int_0^1 \frac{\partial^2 F}{\partial \omega_p \partial \omega_q} \left(x, t \sum_{\substack{k=1 \\ k \neq j}}^m \kappa \phi_k(x + \delta_{kj}) + \kappa \phi_j \right) \sum_{\substack{k=1 \\ k \neq j}}^m \kappa_q \phi_k(x + \delta_{kj}) \chi \, dt \right\|_2 \|\kappa_p w_\xi(x + \delta_{mj})\|_2.$$

But this tends to 0 since the first factor has limit 0 by condition (d) whilst the second factor is bounded. This establishes that $y_i = 0$ for each i .

Now we claim that

$$Lw_\xi + \sum_{p=0}^n \frac{\partial F}{\partial \omega_p} \left(x, \kappa \phi_m \right) \kappa_p w_\xi \rightarrow 0$$

in L^2 . We have by assumption that

$$Lw_\xi + \sum_{p=0}^n \frac{\partial F}{\partial \omega_p} \left(x, \sum_{k=1}^m \kappa \phi_k(x + \delta_{km}) \right) \kappa_p w_\xi \rightarrow 0$$

in L^2 . It suffices therefore to show that

$$\left[\frac{\partial F}{\partial \omega_p} \left(x, \sum_{k=1}^m \kappa \phi_k(x + \delta_{km}) \right) - \frac{\partial F}{\partial \omega_p} \left(x, \kappa \phi_m \right) \right] \kappa_p w_\xi \rightarrow 0$$

in L^2 . The expression on the left-hand side may be written

$$\sum_q \int_0^1 \frac{\partial^2 F}{\partial \omega_p \partial \omega_q} \left(x, t \sum_{k=1}^{m-1} \kappa \phi_k(x + \delta_{km}) + \kappa \phi_m \right) \sum_{k=1}^{m-1} \kappa_q \phi_k(x + \delta_{km}) \kappa_p w_\xi \, dt$$

Dropping the outermost sum and the integral reduces us to showing that

$$\frac{\partial^2 F}{\partial \omega_p \partial \omega_q} \left(x, t \sum_{k=1}^{m-1} \kappa \phi_k(x + \delta_{km}) + \kappa \phi_m \right) \sum_{k=1}^{m-1} \kappa_q \phi_k(x + \delta_{km}) \kappa_p w_\xi$$

tends to 0 in L^2 uniformly for $0 \leq t \leq 1$. Pick out the term

$$\frac{\partial^2 F}{\partial \omega_p \partial \omega_q} \left(x, t \sum_{k=1}^{m-1} \kappa \phi_k(x + \delta_{km}) + \kappa \phi_m \right) \kappa_q \phi_l(x + \delta_{lm}) \kappa_p w_\xi$$

and replace x by $x + \delta_{ml}$. The result is

$$\frac{\partial^2 F}{\partial \omega_p \partial \omega_q} \left(x, t \sum_{k=1}^{m-1} \kappa \phi_k(x + \delta_{kl}) + \kappa \phi_m(x + \delta_{ml}) \right) \kappa_q \phi_l \kappa_p w_\xi(x + \delta_{ml}).$$

This tends to 0 in L^2 uniformly by condition (d) and the fact proved before that the weak limit in H^2 of $w_\xi(x + \delta_{ml})$ is 0.

We therefore have that

$$Lw_\xi + \sum_{p=0}^n \frac{\partial F}{\partial \omega_p} \left(x, \kappa \phi_m \right) \kappa_p w_\xi \rightarrow 0$$

in L^2 . But now the non-degeneracy of the solution ϕ_m implies that $w_\xi \rightarrow 0$ in H^2 contradicting the assumption that $\|w_\xi\|_{2,2} = 1$. Thus condition (5) is verified.

The final step in the proof of the theorem is the verification of condition (6) of Theorem 1.2. The derivative of Γ is given by

$$D_v \Gamma(\xi, v) w = Lw + \sum_{p=0}^n \frac{\partial F}{\partial \omega_p} \left(x, \sum_{k=1}^{m-1} \kappa \phi_k(x + \delta_{km}) + \kappa v \right) \kappa_p w.$$

Hence

$$\begin{aligned} & \left(D_v \Gamma(\xi, v) - D_v \Gamma(\xi, \phi_m) \right) w \\ &= \sum_{p=0}^n \left[\frac{\partial F}{\partial \omega_p} \left(x, \sum_{k=1}^{m-1} \kappa \phi_k(x + \delta_{km}) + \kappa v \right) - \frac{\partial F}{\partial \omega_p} \left(x, \sum_{k=1}^m \kappa \phi_k(x + \delta_{km}) \right) \right] \kappa_p w \\ &= \sum_{p,q} \int_0^1 \frac{\partial^2 F}{\partial \omega_p \partial \omega_q} \left(x, \sum_{k=1}^{m-1} \kappa \phi_k(x + \delta_{km}) + (1-t)\kappa \phi_m + t\kappa v \right) \kappa_p w \kappa_q (v - \phi_m) dt \end{aligned}$$

By the boundedness property of condition (c) the L^2 norm of this quantity is bounded by a constant times $\|v - \phi_m\|_{2,2} \|w\|_{2,2}$ as $\mu(\xi) \rightarrow \infty$ and $v \rightarrow \phi_m$. This implies condition (6) of Theorem 1.2 and concludes the proof of Theorem 2.2.

3 Standard growth conditions

Conditions (c) and (d) preceding Theorem 2.2 both follow from growth conditions on F and its derivatives up to the second order. Define the vector $\Omega = (\omega_1, \dots, \omega_n)$. We consider growth conditions of the following form, where j, k are always distinct from 0:

$$\begin{aligned}
|F(x, \omega_0, \Omega)| &\leq C(|\omega_0| + \|\Omega\| + |\omega_0|^{\alpha_1} + \|\Omega\|^{\beta_1}); \\
\left| \frac{\partial F}{\partial \omega_0}(x, \omega_0, \Omega) \right| &\leq C(1 + |\omega_0|^{\alpha_2} + \|\Omega\|^{\beta_2}); \\
\left| \frac{\partial^2 F}{\partial \omega_0^2}(x, \omega_0, \Omega) \right| &\leq C(1 + |\omega_0|^{\alpha_3} + \|\Omega\|^{\beta_3}); \\
\left| \frac{\partial F}{\partial \omega_k}(x, \omega_0, \Omega) \right| &\leq C(1 + |\omega_0|^{\alpha_4} + \|\Omega\|^{\beta_4}); \\
\left| \frac{\partial^2 F}{\partial \omega_0 \partial \omega_k}(x, \omega_0, \Omega) \right| &\leq C(1 + |\omega_0|^{\alpha_5} + \|\Omega\|^{\beta_5}); \\
\left| \frac{\partial^2 F}{\partial \omega_j \partial \omega_k}(x, \omega_0, \Omega) \right| &\leq C(1 + |\omega_0|^{\alpha_6} + \|\Omega\|^{\beta_6}).
\end{aligned}$$

The constant C is independent of x . The exponents α_i, β_i are non-negative and are chosen so that the following list of functions are all in L^2 whenever $u, v, w \in H^2$, in virtue of the embeddings of H^2 and H^1 into L^p -spaces and Hölder's inequality:

$$\begin{aligned}
&u^{\alpha_1}, \quad \left(\frac{\partial u}{\partial x_i} \right)^{\beta_1}, \quad u^{\alpha_2} v, \quad \left(\frac{\partial u}{\partial x_i} \right)^{\beta_2} v, \\
&u^{\alpha_3} v w, \quad \left(\frac{\partial u}{\partial x_i} \right)^{\beta_3} v w, \quad u^{\alpha_4} \frac{\partial v}{\partial x_k}, \quad \left(\frac{\partial u}{\partial x_i} \right)^{\beta_4} \frac{\partial v}{\partial x_k}, \\
&u^{\alpha_5} v \frac{\partial w}{\partial x_k}, \quad \left(\frac{\partial u}{\partial x_i} \right)^{\beta_5} v \frac{\partial w}{\partial x_k}, \quad u^{\alpha_6} \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial x_k}, \quad \left(\frac{\partial u}{\partial x_i} \right)^{\beta_6} \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial x_k}.
\end{aligned}$$

In addition we must require that the products uv , $u \frac{\partial v}{\partial x_i}$ and $\frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}$ are in L^2 but this is the case if H^1 and H^2 are embedded into L^4 .

Recall here the required Sobolev embeddings (see e.g. [3])

$$H^2 = W^{2,2} \subset L^r$$

where $2 \leq r \leq \infty$ if $n < 4$, $2 \leq r < \infty$ if $n = 4$, $2 \leq r \leq \frac{2n}{n-4}$ if $n > 4$.

$$H^1 = W^{2,1} \subset L^r$$

where $2 \leq r \leq \infty$ if $n < 2$, $2 \leq r < \infty$ if $n = 2$, $2 \leq r \leq \frac{2n}{n-2}$ if $n > 2$.

It is not our object to give an exhaustive treatment here. For our purposes we shall define what we call the *standard conditions*. These are as follows:

Case A. F is independent of ∇u and $1 \leq n \leq 7$. If $1 \leq n \leq 4$ we assume that

$$1 \leq \alpha_1, \quad 0 \leq \alpha_2, \quad 0 \leq \alpha_3$$

with no upper bound. If $n = 5, 6$ or 7 we assume that

$$1 \leq \alpha_1 \leq \frac{n}{n-4}, \quad 0 \leq \alpha_2 \leq \frac{4}{n-4}, \quad 0 \leq \alpha_3 < \frac{8-n}{n-4}$$

Case B. F depends on ∇u and $n = 1, 2$ or 3 . If $n = 1$ or 2 then $\alpha_j, \beta_j, j = 1, \dots, 6$, are only bounded from below by $\alpha_1, \beta_1 \geq 1$ and $\alpha_j, \beta_j \geq 0$ for $j \geq 1$. If $n = 3$ then in addition to these lower bounds we impose the upper bounds $\beta_1 \leq 3, \beta_2 \leq 3, \beta_3 < 3, \beta_4 \leq 2, \beta_5 < 2, \beta_6 < 1$.

Theorem 3.1 *Under the standard conditions the function F satisfies the conditions (c) and (d) of section 2.*

The proof is just a series of applications of the following two lemmas, which in turn are standard applications of Hölder's Inequality and the Rellich-Kondrachov Theorem on compact embeddings of Sobolev spaces [3].

Let V_i denote a family of function spaces drawn from the collection $W^{k,p}(\mathbf{R}^n)$. Let us suppose that V_i is embedded into L^r for $2 \leq r \leq t_i$.

Lemma 3.2 *Let $u_i \in V_i, i = 1, \dots, m$. Then $u_1^{\beta_1} \dots u_m^{\beta_m} \in L^r$ if*

$$\frac{\beta_1}{t_1} + \dots + \frac{\beta_m}{t_m} \leq \frac{1}{r} \leq \frac{\beta_1 + \dots + \beta_m}{2}$$

and

$$\|u_1^{\beta_1} \dots u_m^{\beta_m}\|_r \leq C \|u_1\|_{V_1}^{\beta_1} \dots \|u_m\|_{V_m}^{\beta_m}$$

where C is independent of u_k .

Lemma 3.3 *Suppose that $V_1 = W^{k,p}$ for some $k \geq 1$. Let $w \in V_2$. Then the linear map $v \mapsto vw$ from V_1 to L^r is compact for*

$$\frac{1}{t_1} + \frac{1}{t_2} < \frac{1}{r} \leq 1.$$

Proof of Theorem 3.1

By Lemma 3.2, under the standard conditions all the products listed above belong to L^2 , and condition (c) on the existence and boundedness of \mathbf{F} , \mathbf{F}_i and \mathbf{F}_{ij} is obtained. Consider condition (d) in Case A. We have to show that

$$\int u^{2\alpha_3} v^2 w_\nu^2 dx \rightarrow 0$$

if $w_\nu \rightarrow 0$ weakly in H^2 . If $n = 1, 2, 3, 4$ then by Lemma 3.3 the linear operator $w \mapsto vw$ is compact from H^2 to L^p for $1 \leq p < \infty$. We therefore choose any conjugate exponents r and s and write

$$\int u^{2\alpha_3} v^2 w_\nu^2 dx \leq \left(\int u^{2\alpha_3 r} dx \right)^{\frac{1}{r}} \left(\int (vw_\nu)^{2s} dx \right)^{\frac{1}{s}}$$

and the second factor on the right-hand side tends to 0. If $n = 5, 6, 7$ then the operator $v \mapsto vw$ is compact from H^2 to L^p for $1 \leq p < \frac{n}{n-4}$. We therefore want to choose conjugate exponents r and s so that

$$2\alpha_3 r < \frac{2n}{n-4}, \quad 2s < \frac{n}{n-4}.$$

These are satisfied if we choose r so that

$$\frac{n}{8-n} < r < \frac{n}{n-4} \cdot \frac{1}{\alpha_3}$$

which is allowed in virtue of $\alpha_3 < \frac{8-n}{n-4}$. The convergence to 0 of the right-hand side is clearly uniform w.r.t. holding u in a bounded subset of H^2 .

Let us consider Case B. For $n = 1, 2$ both spaces H^1 and H^2 are embedded in L^p for $2 \leq p < \infty$. So only the case $n = 3$ raises any question and H^2 is still embedded in L^p for $2 \leq p < \infty$. For $n = 3$ the upper bounds for β_k can be read off from Lemma 3.2 and for condition (d) an argument like that above for case A shows that as the upper bounds for β_3 , β_5 and β_6 are strict the necessary compactness obtains. This concludes the proof.

4 Existence of non-degenerate solutions

The main assumption of Theorem 2.2 is the existence of at least one non-degenerate solution. What can be said about the existence and non-degeneracy of solutions?

In this section we examine the problem of finding non-degenerate solutions to the equation $-\Delta u + F(x, u, \nabla u) = 0$ (posed as before in the whole of \mathbf{R}^n) by means of perturbation theory. Throughout this section it will not be assumed that F is periodic in x .

There is an enormous literature on the existence of solutions to the variational problem $-\Delta u + F(x, u) = 0$. The variational approach, by virtue of its great power for proving existence non-constructively, has been extremely popular, and it leads (for second order equations) most naturally to solutions in H^1 , (see the references listed in [15], far from complete). These methods leave unresolved the question of non-degeneracy, indeed they have evolved to avoid this question. It is therefore of interest to point out a conceptually simple approach that always generates non-degenerate solutions in H^2 and that is the case of a small perturbation, with perturbation parameter ϵ , from an autonomous equation, (that is, an equation not explicitly depending on x), corresponding to $\epsilon = 0$.

The method is based on scaling, by which we mean an ϵ -dependent change in the state variables of the problem that becomes singular when $\epsilon = 0$. Scaling was introduced into bifurcation theory by the seminal paper of Crandall and Rabinowitz on bifurcation at a simple eigenvalue [7]. An early example of this method applied to problems of this kind was [11].

In our approach the autonomous equation for $\epsilon = 0$ will have the form $-\Delta u + g(u) = 0$. If ϕ is a solution of this equation the partial derivatives $D_j \phi$ will satisfy

$$(-\Delta + Dg(\phi(x)))D_j \phi(x) = 0.$$

Assuming that the operator $u \mapsto g(u)$ defines a differentiable map of H^2 into L^2 we shall say that the solution $\phi \in H^2$ is *quasi non-degenerate* if the kernel of $-\Delta + Dg(\phi(x)) : H^2 \rightarrow L^2$ is spanned by the n partial derivatives $D_j \phi$, these are linearly independent elements of H^2 , and the range is the orthogonal complement in L^2 of the kernel.

An example of a quasi non-degenerate solution is the so-called ground state solution of $-\Delta u + u - u^p = 0$. Among all non-trivial solutions this is the one for which the energy

$$\int \left(\frac{1}{2} \|\nabla u\|^2 + \frac{1}{2} u^2 - \frac{1}{p+1} u^{p+1} \right) dx$$

is least. The existence and quasi non-degeneracy of the ground state solution are known for all $p > 1$ if $n = 1, 2$, and for $1 < p < \frac{n+2}{n-2}$ if $n \geq 3$. These facts were established over a number of papers, see [14, 18, 9].

We first impose rather sweeping smoothness conditions to enable us to use a simple scaling with a minimum of fuss.

Theorem 4.1 *Let $F_\epsilon(x, u, \nabla u)$ be a smooth function of all its variables (including ϵ), such that it, and its derivatives w.r.t. x and ϵ up to the second order (including the mixed derivative), satisfy the standard growth conditions uniformly w.r.t. ϵ and x . Suppose that $F_0(x, u, \nabla u) = g(u)$ is independent of x and ∇u , and that there exists a quasi non-degenerate solution $\phi \in H^2$ of $-\Delta u + g(u) = 0$. Suppose that $s = c \in \mathbf{R}^n$ is a non-degenerate solution of the equations*

$$\int \left(\frac{\partial}{\partial \epsilon} \right)_{\epsilon=0} F_\epsilon(x - s, \phi(x), \nabla \phi(x)) D_j \phi(x) dx = 0, \quad j = 1, \dots, n. \quad (4.1)$$

Then for all sufficiently small $\epsilon > 0$ the equation $-\Delta u + F_\epsilon(x, u, \nabla u) = 0$ admits a solution $u_\epsilon \in H^2$ which is non-degenerate and approaches $\phi(x + c)$ in the H^2 norm as $\epsilon \rightarrow 0$.

Note. Compare [2], which suggests a similar result in a similar context. The problem is really one of bifurcation from a manifold of solutions, see [17].

Proof Let

$$W = \{w \in H^2 : \int w D_j \phi dx = 0, j = 1, \dots, n\}.$$

We seek a solution to $-\Delta u + F_\epsilon(x, u, \nabla u) = 0$ of the form

$$u(x) = \phi(x + s) + \epsilon w(x + s)$$

where $s \in R^n$ and $w \in W$ are regarded as new variables. The change of “state variable” from u to (s, w) is an example of scaling and is invertible if $\epsilon \neq 0$, since, by the inverse function theorem, (s, w) is uniquely determined in a neighbourhood of $(0, 0)$ by each u in a sufficiently small neighbourhood of ϕ provided $\epsilon \neq 0$.

Substituting into the equation, and using the notation $(\omega_0, \dots, \omega_n) = (u, \nabla u)$ when we are thinking of the variables of F_ϵ , and $\kappa u = (u, \nabla u)$ when we substitute a function, we find after replacing x by $x - s$, that

$$-\epsilon \Delta w + F_\epsilon(x - s, \kappa \phi + \epsilon \kappa w) - F_0(x - s, \kappa \phi) = 0.$$

We divide by ϵ to form the *rescaled problem*

$$-\Delta w + \frac{F_\epsilon(x - s, \kappa \phi + \epsilon \kappa w) - F_0(x - s, \kappa \phi)}{\epsilon} = 0.$$

Letting $\epsilon \rightarrow 0$ we obtain the *limit equation*

$$-\Delta w + Dg(\phi(x))w + \left(\frac{\partial}{\partial \epsilon} \right)_{\epsilon=0} F_\epsilon(x - s, \kappa \phi(x)) = 0.$$

The limit equation has the non-degenerate solution $(s, w) = (c, \eta)$ where η is the unique function in W satisfying

$$-\Delta \eta + Dg(\phi(x))\eta = - \left(\frac{\partial}{\partial \epsilon} \right)_{\epsilon=0} F_\epsilon(x - c, \phi(x), \nabla \phi(x)).$$

We now apply Theorem 1.1, the implicit function theorem; there’s no need for Theorem 1.2 here. We obtain a non-degenerate solution (s, w) to the rescaled equations and hence a non-degenerate solution u_ϵ to $Lu + F_\epsilon(x, u, \nabla u) = 0$ for all sufficiently small ϵ .

We should check that the limits occur in the right topology. The derivative of the rescaled equation w.r.t. (s, w) is the linear map from $\mathbf{R}^n \times W$ to L^2 given by

$$(\sigma, z) \mapsto -\Delta z + \sum_{i=0}^n \frac{\partial F_\epsilon}{\partial \omega_i} (x - s, \kappa\phi + \epsilon\kappa w) \kappa_i z - \sum_{i=1}^n \frac{1}{\epsilon} \frac{\partial F_\epsilon}{\partial x_i} (x - s, \kappa\phi + \epsilon\kappa w) \sigma_i.$$

The assumption that $\frac{\partial F_\epsilon}{\partial x_i}$ satisfies the standard conditions implies that this derivative exists. Checking condition (5) of Theorem 1.1 boils down to showing two things: firstly that

$$\frac{1}{\epsilon} \frac{\partial F_\epsilon}{\partial x_i} (x - s, \kappa\phi + \epsilon\kappa w)$$

converges in L^2 as $\epsilon \rightarrow 0$, $s \rightarrow c$, $w \rightarrow \eta$; and secondly that the linear map

$$z \mapsto \frac{\partial F_\epsilon}{\partial \omega_i} (x - s, \kappa\phi + \epsilon\kappa w) \kappa_i z$$

converges in the operator norm topology in the space $\mathcal{L}(H^2, L^2)$ as $\epsilon \rightarrow 0$, $s \rightarrow c$, $w \rightarrow \eta$. These are guaranteed by the assumption that the standard growth conditions hold for F_ϵ , $\frac{\partial F_\epsilon}{\partial x_i}$ and $\frac{\partial}{\partial \epsilon} \frac{\partial F_\epsilon}{\partial x_i}$ uniformly w.r.t. ϵ .

Since the implicit function theorem of necessity gives us solutions of the rescaled equation that are non-degenerate, the corresponding solutions of the original problem are also non-degenerate, except of course for $\epsilon = 0$. This ends the proof.

Note that we get more than was stated in the theorem; the solutions form a continuous curve parametrized by ϵ and we have the limits that s and w tend to; in particular the limit of w is explicit and non-zero in general.

We can exhibit a class of problems in which the existence of a non-degenerate solution to equation (4.1) of Theorem 4.1 is guaranteed by the non-vanishing of a single integral.

Theorem 4.2 *Under the conditions of Theorem 4.1 let*

$$H(x, \omega_0, \Omega) = \left(\frac{\partial}{\partial \epsilon} \right)_{\epsilon=0} F_\epsilon(x, \omega_0, \Omega).$$

Assume that the functions $H(\cdot, \omega_0, \Omega) : \mathbf{R}^n \rightarrow \mathbf{R}$, $H(x, \omega_0, \cdot) : \mathbf{R}^n \rightarrow \mathbf{R}$ and ϕ are invariant under reflection in the coordinate hyperplanes and permutation of coordinates. Assume that

$$\int \left(H(x, \kappa\phi) \Delta \phi + \frac{\partial H}{\partial \omega_0} (x, \kappa\phi) \|\nabla \phi\|^2 + \frac{1}{2} \sum_{k=1}^n \frac{\partial H}{\partial \omega_k} (x, \kappa\phi) \frac{\partial}{\partial x_k} \|\nabla \phi\|^2 \right) dx \neq 0.$$

Then $s = 0$ is a non-degenerate solution of (4.1).

Proof Define

$$K_j(s) = \int H(x - s, \phi, \nabla\phi) D_j\phi \, dx.$$

Let τ_{ij} interchange the i th and j th coordinates and let ρ_i switch the sign of the i th coordinate. Letting ρ_j act on the variable of integration gives

$$\int H(x, \phi, \nabla\phi) D_j\phi \, dx = - \int H(x, \phi, \nabla\phi) D_j\phi \, dx.$$

Hence $s = 0$ is a solution of (4.1). We write

$$\int H(x - s, \phi, \nabla\phi) D_j\phi \, dx = \int H(x, \phi(x + s), \nabla\phi(x + s)) D_j\phi(x + s) \, dx$$

and derive the partial derivative $\partial K_j / \partial s_i$ at $s = 0$

$$\begin{aligned} \frac{\partial K_j}{\partial s_i} = \int & \left(H(x, \phi, \nabla\phi) D_{i,j}\phi + \frac{\partial H}{\partial \omega_0}(x, \phi, \nabla\phi) D_i\phi D_j\phi \right. \\ & \left. + \frac{1}{2} \sum_{k=1}^n \frac{\partial H}{\partial \omega_k}(x, \phi, \nabla\phi) \frac{\partial}{\partial x_k} (D_i\phi)^2 \right) dx. \end{aligned}$$

If $i \neq j$ this integral is 0, as may be seen by letting ρ_i or ρ_j act on the variable of integration. Letting τ_{ij} act on the variable of integration we see that $\partial K_i / \partial s_i$ is independent of i and summing over i we obtain the proclaimed integral.

5 More scalings

The scaling used in the proof of Theorem 4.1, involving the replacement of x by $x - s$, is not appropriate if F_ϵ is not a differentiable function of x , as the rescaled equation is then not a differentiable function of s . This case is important in applications as we may wish to handle polynomials in u whose coefficients are measurable functions of x .

Note that the finite-dimensional problem satisfied by c may be written

$$\int \left(\frac{\partial}{\partial \epsilon} \right)_{\epsilon=0} F_\epsilon(x, \phi(x + s), \nabla\phi(x + s)) D_j\phi(x + s) \, dx = 0, \quad j = 1, \dots, n \quad (5.1)$$

and is still differentiable w.r.t. s though F_ϵ is not differentiable w.r.t. x . In fact the differentiability of F_ϵ w.r.t. x is unnecessary and the argument in the proof of Theorem 4.1 can be carried through without the shift. One simply writes

$$u(x) = \phi(x + s) + \epsilon w(x + c)$$

where $w \in W$, and in the place where we replaced x by $x - s$ we instead replace it by $x - c$. This leads to the rescaled equation

$$-\Delta w + \frac{F_\epsilon(x - c, \kappa\phi(x + s - c) + \epsilon\kappa w) - F_0(x - c, \kappa\phi(x + s - c))}{\epsilon} = 0 \quad (5.2)$$

with limit equation

$$-\Delta w + Dg(\phi(x + s - c))w + \left(\frac{\partial}{\partial \epsilon}\right)_{\epsilon=0} F_\epsilon(x - c, \kappa\phi(x + s - c)) = 0 \quad (5.3)$$

thus alleviating the need to assume $F_\epsilon(x, u, \nabla u)$ to be a differentiable function of x . Note that the map $s \mapsto \phi(x + s - c)$, from \mathbf{R}^n to H^2 is differentiable since the assumption of quasi non-degeneracy implies that $D_i\phi \in H^2$.

Equation (5.2) raises some interesting questions which have not yet been treated in a systematic fashion and are mentioned here in passing. For example to solve (5.2) we don't really need differentiability w.r.t. ϵ , we only need the convergence of the quotient as $\epsilon \rightarrow 0$ together with that of its derivative w.r.t. (s, w) . Moreover there are cases where the quotient does not converge regarded as an operator from H^2 to L^2 but does so when regarded as acting from H^1 to H^{-1} . Some special cases of this were treated in [11]. More recent results in this direction may be seen in [4].

There is however another approach; that is to avoid the use of any shift that is s -dependent but allow one that is ϵ -dependent only. This can be used to obtain more precise asymptotic information, but at the expense of more complex calculations. We continue to avoid the assumption that F_ϵ is a differentiable function of x but we now suppose that F_ϵ does not depend on the gradient ∇u . We also assume that the second order derivatives of ϕ belong to H^2 . As for growth conditions we assume that F_ϵ , $\frac{\partial}{\partial \epsilon} F_\epsilon$ and $\frac{\partial^2}{\partial \epsilon^2} F_\epsilon$ all satisfy the standard conditions.

For simplicity we suppose that c is the 0 vector; that is, $s = 0$ is a non-degenerate solution of the equations (5.1). Recall that $F_0(x, u)$ is the function $g(u)$. We use the scaling

$$u = \phi(x + c_\epsilon) + \epsilon v_\epsilon(x + c_\epsilon) + \epsilon\beta \cdot \nabla\phi(x + c_\epsilon) + \epsilon^2 w(x + c_\epsilon)$$

where $w \in W$. The new variables are $\beta \in \mathbf{R}^n$ and $w \in W$, whilst $c_\epsilon \in \mathbf{R}^n$ and $v_\epsilon \in W$ are functions of ϵ to be specified presently.

Since $s = 0$ is a non-degenerate solution of the equations

$$\int \left(\frac{\partial}{\partial \epsilon}\right)_{\epsilon=0} F_\epsilon(x, \phi(x + s)) D_j \phi(x + s) dx = 0, \quad j = 1, \dots, n$$

the implicit function theorem gives $c_\epsilon \in \mathbf{R}^n$ depending differentiably on ϵ such that $c_0 = 0$ and

$$\int \frac{F_\epsilon(x, \phi(x + c_\epsilon)) - F_0(x, \phi(x + c_\epsilon))}{\epsilon} D_j \phi(x + c_\epsilon) dx = 0, \quad j = 1, \dots, n$$

or, equivalently

$$\int \frac{F_\epsilon(x - c_\epsilon, \phi(x)) - F_0(x - c_\epsilon, \phi(x))}{\epsilon} D_j \phi(x) dx = 0, \quad j = 1, \dots, n.$$

Hence there exists a unique $v_\epsilon \in W$ such that

$$-\Delta v_\epsilon + Dg(\phi(x))v_\epsilon = -\frac{F_\epsilon(x - c_\epsilon, \phi(x)) - F_0(x - c_\epsilon, \phi(x))}{\epsilon}$$

and

$$-\Delta v_0 + Dg(\phi(x))v_0 = -\left(\frac{\partial}{\partial \epsilon}\right)_{\epsilon=0} F_\epsilon(x, \phi(x)).$$

Substitute the scaling into $-\Delta u + F_\epsilon(x, u) = 0$, replace x by $x - c_\epsilon$, use the equations satisfied by $\phi(x)$, $\nabla \phi$ and v_ϵ , and finally divide by ϵ^2 . We obtain the *rescaled equation*, which we write in expanded form

$$\begin{aligned} -\Delta w + \frac{1}{\epsilon} \left[\frac{\partial F_\epsilon}{\partial u}(x - c_\epsilon, \phi) - \frac{\partial F_0}{\partial u}(x - c_\epsilon, \phi) \right] (v_\epsilon + \beta \cdot \nabla \phi) \\ + \frac{1}{\epsilon^2} \left[F_\epsilon(x - c_\epsilon, \phi + \epsilon(v_\epsilon + \beta \cdot \nabla \phi) + \epsilon^2 w) - F_\epsilon(x - c_\epsilon, \phi) \right. \\ \left. - \epsilon \frac{\partial F_\epsilon}{\partial u}(x - c_\epsilon, \phi)(v_\epsilon + \beta \cdot \nabla \phi) \right] = 0. \end{aligned} \quad (5.4)$$

The *limit equation* is

$$-\Delta w + Dg(\phi)w + \frac{1}{2}D^2g(\phi)(v_0 + \beta \cdot \nabla \phi)^2 + \left(\frac{\partial}{\partial \epsilon}\right)_{\epsilon=0} \frac{\partial F_\epsilon}{\partial u}(x, \phi)(v_0 + \beta \cdot \nabla \phi) = 0.$$

We solve the limit equation by the Fredholm alternative. First we need to find β satisfying

$$\int \left(\frac{1}{2}D^2g(\phi)(v_0 + \beta \cdot \nabla \phi)^2 + \left(\frac{\partial}{\partial \epsilon}\right)_{\epsilon=0} \frac{\partial F_\epsilon}{\partial u}(x, \phi)(v_0 + \beta \cdot \nabla \phi) \right) D_j \phi dx = 0$$

for $j = 1, \dots, n$. This apparently quadratic system in the vector β is actually linear. To see this we differentiate the equation $-\Delta \phi + g(\phi) = 0$ twice. This gives

$$-\Delta D_{i,j} \phi + Dg(\phi)D_{i,j} \phi + D^2g(\phi)D_i \phi D_j \phi = 0 \quad (5.5)$$

As, by assumption, $D_{i,j} \phi \in H^2$, we find that

$$\int D^2g(\phi)D_i \phi D_j \phi D_k \phi dx = 0$$

which implies that the quadratic terms in β vanish. The equation for β therefore takes the form

$$A\beta + \gamma = 0$$

for a certain matrix A and vector γ . But in fact A is non-singular, being the jacobian at $s = 0$ of the mapping $s \mapsto K(s) = (K_1, \dots, K_n)(s)$, from \mathbf{R}^n to itself, where

$$K_i(s) = \int \left(\frac{\partial}{\partial \epsilon} \right)_{\epsilon=0} F_\epsilon(x, \phi(x+s)) D_i \phi(x+s) dx = 0.$$

The jacobian is non-singular since $s = 0$ was assumed to be a non-degenerate solution of the equations (5.1). To see that A is indeed the jacobian we first compute

$$A_{ij} = \int \left(D^2 g(\phi) v_0 + \left(\frac{\partial}{\partial \epsilon} \right)_{\epsilon=0} \frac{\partial F_\epsilon}{\partial u}(x, \phi) \right) D_i \phi D_j \phi dx.$$

Now we have from (5.5) that

$$\int \left(v_0 \left(-\Delta + Dg(\phi) \right) D_{i,j} \phi + D^2 g(\phi) v_0 D_i \phi D_j \phi \right) dx = 0,$$

by symmetry of $-\Delta$ this is

$$\int \left(D_{i,j} \phi \left(-\Delta + Dg(\phi) \right) v_0 + D^2 g(\phi) v_0 D_i \phi D_j \phi \right) dx = 0,$$

and by the equation satisfied by v_0 we have

$$\int \left(-D_{i,j} \phi \left(\frac{\partial}{\partial \epsilon} \right)_{\epsilon=0} F_\epsilon(x, \phi) + D^2 g(\phi) v_0 D_i \phi D_j \phi \right) dx = 0.$$

Hence we have

$$A_{ij} = \int \left(D_{i,j} \phi \left(\frac{\partial}{\partial \epsilon} \right)_{\epsilon=0} F_\epsilon(x, \phi) + \left(\frac{\partial}{\partial \epsilon} \right)_{\epsilon=0} \frac{\partial F_\epsilon}{\partial u}(x, \phi) D_i \phi D_j \phi \right) dx = \left. \frac{\partial K_i}{\partial s_j} \right|_{s=0}$$

The limit equation has therefore a non-degenerate solution β_0, w_0 say, and we obtain a “principal part” (non-zero limiting values for β and w).

To check that we may apply the implicit function theorem in the form of Theorem 1.1 we have to study more carefully the approach of the rescaled problem (5.4) to the limit equation. This requires checking that:

1) The operator

$$z \mapsto \frac{\partial F_\epsilon}{\partial u} \left(x - c_\epsilon, \phi + \epsilon(v_\epsilon + \beta \cdot \nabla \phi) + \epsilon^2 w \right) z$$

converges in the operator norm as $\epsilon \rightarrow 0$, $\beta \rightarrow \beta_0$, $w \rightarrow w_0$, to the operator

$$z \mapsto \frac{\partial g}{\partial u}(x, \phi)z.$$

2) The function

$$\frac{1}{\epsilon} \left[\frac{\partial F_\epsilon}{\partial u}(x - c_\epsilon, \phi + \epsilon(v_\epsilon + \beta \cdot \nabla \phi) + \epsilon^2 w) - \frac{\partial F_0}{\partial u}(x, \phi) \right] D_j \phi$$

converges in L^2 as $\epsilon \rightarrow 0$, $\beta \rightarrow \beta_0$, $w \rightarrow w_0$, to

$$\left(\frac{\partial}{\partial \epsilon} \right)_{\epsilon=0} \frac{\partial F_\epsilon}{\partial u}(x, \phi) D_j \phi$$

These are guaranteed by the assumption that $F_\epsilon(x, u)$ and its derivatives w.r.t. ϵ up to the second order satisfy the standard growth conditions.

6 Positivity

In this section we return to the problem considered in section 2 but restrict ourselves to the special case

$$-\Delta u + F(x, u) = 0, \tag{6.1}$$

our object being to show that the solutions established in Theorem 2.2 are positive under certain conditions. We assume that $F(x, u)$ is periodic in x with period lattice Λ and that it satisfies conditions (a), (b), (c) and (d) of section 2.

Let ϕ_k , $k = 1, \dots, m$, be non-degenerate solutions in $H^2(\mathbf{R}^n)$ to (6.1). Theorem 2.2 shows the existence of a solution u_ϵ near to $\sum_{k=1}^m \phi_k(x + \xi_k)$ provided $\mu(\xi)$ is large enough.

Let

$$G(x, u) = \frac{F(x, u)}{u}$$

Note that

$$G(x, u) = \int_0^1 \frac{\partial F}{\partial u}(x, tu) dt.$$

and that $G(x, 0) = \frac{\partial F}{\partial u}(x, 0)$. We make the following additional assumptions:

- e) There exist $h > 0$ and $H > 0$ such that $h < \frac{\partial F}{\partial u}(x, 0) < H$ for all $x \in \mathbf{R}^n$.
- f) For each k the function ϕ_k is strictly positive.

Note that $-\Delta + G(x, \phi_k(x))$ is a self-adjoint operator in L^2 and is a compact perturbation of the positive operator $-\Delta + \frac{\partial F}{\partial u}(x, 0)$ by condition (d) of section 2. Under condition (f)

standard theory [13] shows that the function ϕ_k is the ground state of the Hamiltonian $S_k = -\Delta + G(x, \phi_k(x))$. Moreover S_k has one-dimensional kernel spanned by ϕ_k and no negative eigenvalues.

Theorem 6.1 *Let F satisfy the conditions of Theorem 2.2 with the restrictions that $L = -\Delta$ and F is independent of ∇u . In addition assume (e) and (f). Then the solution $u_\xi(x)$ given by Theorem 2.2 is strictly positive for all sufficiently large $\mu(\xi)$.*

Note. In common with many variational approaches [6] contains a discussion about positive solutions, which has a totally different character from what we present here.

Proof Our object is to show that u_ξ is the ground state of the linear Schrödinger operator $T_\xi = -\Delta + G(x, u_\xi(x))$ for sufficiently large $\mu(\xi)$; that is, since $T_\xi u_\xi = 0$, we must show that the operator T_ξ has no negative eigenvalues. Standard results [13] then imply that u_ξ is strictly of one sign, and is therefore positive since it approximates the positive function $\sum \phi_k(x + \xi_k)$.

Lemma 6.2 *There exists an interval $]-\rho_0, \rho_0[$ such that for sufficiently large $\mu(\xi)$ the operator T_ξ has no eigenvalue other than 0 in the interval $]-\rho_0, \rho_0[$ and this eigenvalue is simple.*

Proof Let

$$Z_\xi = \{v \in H^2 : \int v(x)u_\xi(x) dx = 0\}$$

and define the linear operator $S_\xi : \mathbf{R} \times Z_\xi \rightarrow L^2$ by

$$S_\xi(s, z) = su_\xi + T_\xi z.$$

The main part of the proof of the lemma consists in proving that S_ξ^{-1} exists and $\|S_\xi^{-1}\|$ has a uniform bound for $\mu(\xi)$ sufficiently large. Indeed suppose on the contrary that this is not the case. Then there exists an admissible set $X \subset \Lambda$ and for each $\xi \in X$ there exists $(s_\xi, z_\xi) \in \mathbf{R} \times Z_\xi$ such that

$$|s_\xi| + \|z_\xi\|_{2,2} = 1, \quad \text{whilst} \quad \lim_{\mu(\xi) \rightarrow \infty, \xi \in X} \|S_\xi(s_\xi, z_\xi)\|_2 = 0$$

that is, we have

$$-\Delta z_\xi + G(x, u_\xi(x))z_\xi + s_\xi u_\xi \rightarrow 0$$

in L^2 . We may assume (replacing X by a smaller admissible set if necessary) that

$$\lim_{\mu(\xi) \rightarrow \infty} s_\xi = s_\infty, \quad \lim_{\mu(\xi) \rightarrow \infty} z_\xi(x - \xi_k) = y_k(x)$$

weakly in H^2 . Replacing x by $x - \xi_k$ and going to the distribution limit (cf. the calculations of section 2), we find that

$$-\Delta y_k + G(x, \phi_k(x))y_k + s_\infty \phi_k(x) = 0$$

and $\int y_k(x)\phi_k(x) dx = 0$. We deduce by condition (f) that $s_\infty = 0$ and $y_k = 0$.

Now we claim that

$$\left(G(x, u_\xi) - \frac{\partial F}{\partial u}(x, 0)\right)z_\xi \rightarrow 0$$

in the L^2 -norm. We may write this expression as

$$\int_S \frac{\partial^2 F}{\partial u^2}(x, stu_\xi)tu_\xi z_\xi ds dt.$$

To prove the claim it is therefore enough to show that

$$\frac{\partial^2 F}{\partial u^2}(x, tu_\xi)u_\xi z_\xi \rightarrow 0$$

in L^2 uniformly for $0 \leq t \leq 1$. Decompose the solution $u_\xi(x) = \sum_{k=1}^m u_{\xi,k}(x - \xi_k) \rightarrow \phi_k$ in the H^2 norm. Since, as shown, $z_\xi(x - \xi_k) \rightarrow 0$ weakly in H^2 we obtain (by condition (d), section 2)

$$\frac{\partial^2 F}{\partial u^2}(x, tu_\xi(x - \xi_k))u_{\xi,k}(x - \xi_k)z_\xi(x - \xi_k) \rightarrow 0$$

in the L^2 norm uniformly for $0 \leq t \leq 1$. Hence also

$$\frac{\partial^2 F}{\partial u^2}(x, tu_\xi)u_{\xi,k}z_\xi \rightarrow 0$$

in L^2 uniformly and summing over k we arrived at the claimed result.

We therefore see that

$$-\Delta z_\xi + \frac{\partial F}{\partial u}(x, 0)z_\xi \rightarrow 0$$

in the L^2 norm. By condition (e) and Wang's Lemma (see next section) this yields $\|z_\xi\|_{2,2} \rightarrow 0$, which, together with $|s_\xi| \rightarrow 0$ contradicts our assumptions about (s_ξ, z_ξ) .

To finish the proof of the lemma we choose $\rho_0 > 0$ such that the operator $(s, z) \mapsto S_\xi(s, z) - \lambda z$, from $\mathbf{R} \times Z_\xi$ to L^2 , is invertible for $|\lambda| < \rho_0$ and $\mu(\xi)$ sufficiently large. This is possible because, as just shown, $\|S_\xi^{-1}\|$ is uniformly bounded for large $\mu(\xi)$. Suppose that λ is an eigenvalue of T_ξ for which $|\lambda| < \rho_0$. Let $v \neq 0$ be the eigenfunction and write $v = su_\xi + z$ where $z \in Z_\xi$. Then we have

$$S_\xi(-\lambda s, z) - \lambda z = 0$$

whence $(-\lambda s, z) = (0, 0)$ so that $z = 0$ and $\lambda = 0$. This proves the lemma.

We now complete the proof of Theorem 6.1. Suppose, to the contrary, that T_ξ has a negative eigenvalue $\lambda_\xi < 0$ for each ξ in an admissible set $X \subset \Lambda$, with corresponding eigenfunction v_ξ where $\|v_\xi\|_{2,2} = 1$. Then

$$-\Delta v_\xi + G(x, u_\xi(x))v_\xi = \lambda_\xi v_\xi.$$

We may assume that $\lambda_\xi \rightarrow \lambda_\infty$, and by Lemma 6.2, $\lambda_\infty < 0$ with possibly $\lambda_\infty = -\infty$. We may assume that

$$v_\xi(x - \xi_k) \rightarrow w_k$$

weakly in H^2 and we deduce, by shifting and going to the distribution limit, that

$$-\Delta w_k + G(x, \phi_k(x))w_k = \lambda_\infty w_k$$

provided $\lambda_\infty \neq -\infty$. Hence, by condition (f), $w_k = 0$. If $\lambda_\infty = -\infty$ we still have that $\lambda_\xi v_\xi(x - \xi_k)$ has a finite limit, and since $v_\xi(x - \xi_k)$ tends to w_k we again have that $w_k = 0$. By the same argument as in the proof of Lemma 6.2 we now deduce that

$$\left(G(x, u_\xi) - \frac{\partial F}{\partial u}(x, 0) \right) v_\xi \rightarrow 0$$

in L^2 and hence

$$-\Delta v_\xi + \frac{\partial F}{\partial u}(x, 0)v_\xi - \lambda_\xi v_\xi \rightarrow 0$$

in L^2 . By condition (f), and the fact that $-\infty < \lambda_\xi < 0$, Wang's Lemma implies the contradiction $\|v_\xi\|_{2,2} \rightarrow 0$.

7 Wang's Lemma

In [16] X. Wang showed the following:

Theorem 7.1 *Let $V(x)$ be a measurable function on \mathbf{R}^n such that $0 < h < V(x) < K$ for some constants h and K . Then*

$$\| -\Delta v + V(\epsilon x)v \|_2 \geq C \|v\|_{2,2}$$

where C is independent of ϵ .

This is a case of a more general result, which will prove widely useful, and which we continue to refer to as Wang's Lemma.

Theorem 7.2 *Let H be a Hilbert space and A_ν a sequence of unbounded self-adjoint operators on H with common domain $D \subset H$. Let B be a positive, self-adjoint operator on H with domain D and assume the following:*

- (1) *$d(0, \text{sp}(A_\nu)) > h > 0$ for some constant h independent of ν ;*
- (2) *there exist K independent of ν and a family of non-negative numbers μ_ν such that $\|A_\nu x - Bx - \mu_\nu x\| \leq K\|x\|$ for all $x \in D$.*

Suppose that $x_\nu \in D$ satisfies $\|A_\nu x_\nu\| \rightarrow 0$. Then $\|x_\nu\| + \|Bx_\nu\| \rightarrow 0$.

The conclusion can be phrased as follows: the norm of the operator A_ν^{-1} from H to the graph norm topology on D determined by B is uniformly bounded.

Clearly Wang's result follows from Theorem 7.2 by writing $H = L^2$, $D = H^2$, $A_\nu = -\Delta + V(\epsilon_\nu x)$ and $B = -\Delta$. I am indebted to C. A. Stuart for helping me to see that Wang's Lemma could be formulated in this useful fashion.

Proof By spectral theory $\|A_\nu^{-1}\| \leq 1/h$. Hence

$$\|x_\nu\| \leq \|A_\nu^{-1}\| \cdot \|A_\nu x_\nu\| \leq \frac{1}{h} \|A_\nu x_\nu\| \rightarrow 0.$$

We therefore also have

$$\|Bx_\nu + \mu_\nu x_\nu\| \leq \|A_\nu x_\nu - Bx_\nu - \mu_\nu x_\nu\| + \|A_\nu x_\nu\| \leq K\|x_\nu\| + \|A_\nu x_\nu\| \rightarrow 0.$$

Hence $\|Bx_\nu + \mu_\nu x_\nu\|^2 \rightarrow 0$ and so

$$\|Bx_\nu\|^2 + 2\mu_\nu \langle Bx_\nu, x_\nu \rangle + \mu_\nu^2 \|x_\nu\|^2 \rightarrow 0$$

and, all terms on the left-hand side being positive, this gives $\|Bx_\nu\| \rightarrow 0$. The conclusion follows.

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