Viðauki D

HNITASKIPTI

D.1 Hornrétt hnitaskipti

Oft er mikilvægt að sjá hvernig hlutafleiðuvirkjar breytast þegar skipt er um hnit. Við ætlum nú að athuga hvernig virkjarnir grad, div, rot og Δ breytast við hnitaskipti. Þessir virkjar eru skrifaðir með ýmsum ritþætti,

$$\text{grad} = \nabla = \partial, \qquad \text{div} = \nabla\cdot, \qquad \text{rot} = \text{curl} = \nabla\times, \qquad \Delta = \nabla\cdot\nabla = \nabla^2.$$

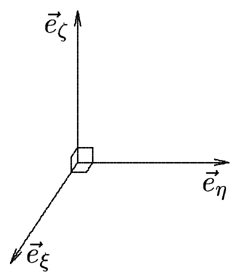
Í þessum kafla skulum við nota táknin ∇, $\nabla\cdot$, $\nabla\times$ og ∇^2 eins og viðtekið er í eðlisfræði. Hugsum okkur að U sé opið mengi í (x, y, z)-rúmi, að vörpunin

$$\vec{r} : V \to U, \qquad (\xi, \eta, \zeta) \mapsto (x, y, z) = (x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta))$$

sé tvisvar samfellt deildanleg og lítum á (ξ, η, ζ) sem ný hnit á U. Við gerum ráð fyrir að þau séu hornrétt, en það þýðir að vigrarnir

$$\vec{a} = \frac{\partial \vec{r}}{\partial \xi} = \left(\frac{\partial x}{\partial \xi}, \frac{\partial y}{\partial \xi}, \frac{\partial z}{\partial \xi}\right), \quad \vec{b} = \frac{\partial \vec{r}}{\partial \eta} \quad \text{og} \quad \vec{c} = \frac{\partial \vec{r}}{\partial \zeta}$$

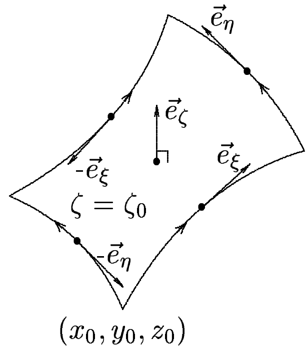
eru hornréttir í sérhverjum punkti (ξ, η, ζ). Við táknum lengdir þeirra með a, b og c og einingarvigra í stefnu þeirra með $\vec{e}_\xi$, $\vec{e}_\eta$ og $\vec{e}_\zeta$. Við skulum einnig gefa okkur að áttun $\vec{e}_\xi$, $\vec{e}_\eta$ og $\vec{e}_\zeta$ í þessari röð sé þannig, að þeir myndi hægri handar kerfi.



Mynd D.1. Hægri handar kerfi.

Ferill sem stikaður er með $\xi \mapsto \vec{r}(\xi, \eta_0, \zeta_0)$ nefnist ξ-hnitaferill og það er ljóst að $\partial_\xi \vec{r}(\xi, \eta_0, \zeta_0)$ er snetill við hann. Við skilgreinum η- og ζ-hnitaferla með hliðstæðum hætti og sjáum að $\partial_\eta \vec{r}(\xi_0, \eta, \zeta_0)$ og $\partial_\zeta \vec{r}(\xi_0, \eta_0, \zeta)$ eru snertlar við þá. Bogalengdarfrymin á ξ-, η- og ζ-hnitaferlunum eru því

$$ds_\xi = a\,d\xi, \qquad ds_\eta = b\,d\eta \qquad \text{og} \qquad ds_\zeta = c\,d\zeta.$$



Mynd D.2. Snertlar við hnitaferla.

Hnitaflötur gegnum punktinn (x_0, y_0, z_0) er myndmengi af plani í gegnum $\xi_0, \eta_0, \zeta_0)$ þar sem einu hniti er haldið föstu. Til dæmis er ζ-hnitaaflötur stikaður með $(\xi, \eta) \mapsto \vec{r}(\xi, \eta, \zeta_0)$. Flatarmálsfrymið á þessum hnitafleti er

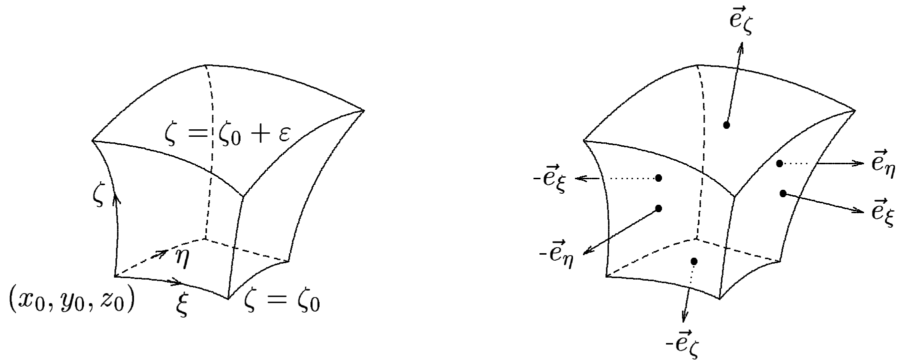
$$dS_\zeta = \|\frac{\partial \vec{r}}{\partial \xi} \times \frac{\partial \vec{r}}{\partial \eta}\| d\xi d\eta.$$

Fyrst vigrarnir $\partial_\xi \vec{r}$, $\partial_\eta \vec{r}$ og $\partial_\zeta \vec{r}$ eru hornréttir, þá er lengdin af krossfeldi þeirra jöfn margfeldi lengdanna ab. Á hliðstæðan hátt fást formúlur fyrir flatarmálsfrymin á ξ- og η-hnitaflötunum og niðurstaðan verður

$$dS_\xi = bc\, d\eta d\zeta, \qquad dS_\eta = ac\, d\xi d\zeta \qquad \text{og} \qquad dS_\zeta = ab\, d\xi d\eta.$$

Vigrarnir $\partial_\xi \vec{r}$, $\partial_\eta \vec{r}$ og $\partial_\zeta \vec{r}$ mynda dálkana í Jacobi-fylki vörpunarinnar $\vec{r}$. Fyrst þeir eru hornréttir, þá er tölugildið af Jacobi-ákveðu hnitaskiptanna jafnt abc og rúmmálsfrymið verður,

$$dV = dxdydz = abc\, d\xi d\eta d\zeta.$$



Mynd D.3. Rúmskiki í (x, y, z)-rúmi.

D.2 Stigull í pólhnitum og kúluhnitum

Nú skulum við líta á fall $f : U \to \mathbb{R}$. Í rétthyrndu hnitakerfi er stigullinn af f settur fram sem

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \frac{\partial f}{\partial x}\vec{e}_x + \frac{\partial f}{\partial y}\vec{e}_y + \frac{\partial f}{\partial z}\vec{e}_z,$$

þar sem $\vec{e}_x = (1, 0, 0)$, $\vec{e}_y = (0, 1, 0)$ og $\vec{e}_z = (0, 0, 1)$. Nú setjum við $\varphi(\xi, \eta, \zeta) = f(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta))$. Keðjureglan gefur okkur formúluna

(D.2.1) $$\nabla f = \frac{1}{a}\frac{\partial \varphi}{\partial \xi}\vec{e}_\xi + \frac{1}{b}\frac{\partial \varphi}{\partial \eta}\vec{e}_\eta + \frac{1}{c}\frac{\partial \varphi}{\partial \zeta}\vec{e}_\zeta,$$

þar sem litið er á vinstri hliðina sem fall af (x, y, z) og hægri hliðina sem fall af (ξ, η, ζ). Í sértilfellinu þegar f er fall af tveimur breytistærðum og hnitaskiptin eru þannig að z hnitið er óbreytt við hnitaskiptin,

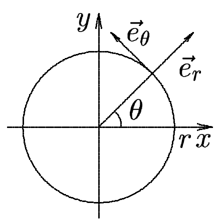
$$(\xi, \eta, \zeta) \mapsto (x, y, z) = (x(\xi, \eta), y(\xi, \eta), \zeta),$$

þá verður þessi formúla

$$\nabla f = \frac{1}{a}\frac{\partial \varphi}{\partial \xi}\vec{e}_\xi + \frac{1}{b}\frac{\partial \varphi}{\partial \eta}\vec{e}_\eta. \tag{D.2.2}$$

Sýnidæmi D.2.1 (*Stigull í pólhnitum*). Við skulum nú innleiða pólhnit, $(x, y) = (r\cos\theta, r\sin\theta)$. Þá er

$$\begin{aligned}
\vec{a} &= \left(\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}\right) = \big(\cos\theta, \sin\theta\big), & a &= 1, & \vec{e}_r &= \big(\cos\theta, \sin\theta\big),\\
\vec{b} &= \left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}\right) = \big(-r\sin\theta, r\cos\theta\big), & b &= r, & \vec{e}_\theta &= \big(-\sin\theta, \cos\theta\big).
\end{aligned}$$



Mynd D.4. Pólhnit.

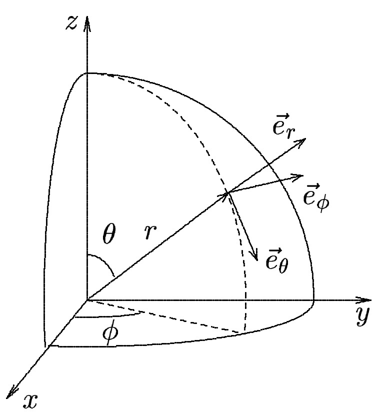
Stigullinn í pólhnitum er því

$$\nabla f = \frac{\partial \varphi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial \varphi}{\partial \theta}\vec{e}_\theta. \tag{D.2.3}$$

□

Sýnidæmi D.2.2 Stigull í kúluhnitum Við skulum nú innleiða kúluhnit,

$$(x, y, z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).$$



Mynd D.5. Kúluhnit.

Þá er

$$\begin{aligned}
\vec{a} &= \left(\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r}\right) = \big(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\big), & a &= 1, \\
\vec{b} &= \left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta}\right) = \big(r\cos\theta\cos\phi, r\cos\theta\sin\phi, -r\sin\theta\big), & b &= r, \\
\vec{c} &= \left(\frac{\partial x}{\partial \phi}, \frac{\partial y}{\partial \phi}, \frac{\partial z}{\partial \phi}\right) = \big(-r\sin\theta\sin\phi, r\sin\theta\cos\phi, 0), & c &= r\sin\theta.
\end{aligned}$$

Grunnurinn í kúluhnitum er því

$$\begin{aligned}
\vec{e}_r &= \big(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\big), \\
\vec{e}_\theta &= \big(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta\big), \\
\vec{e}_\phi &= \big(-\sin\phi, \cos\phi, 0).
\end{aligned}$$

Stigullinn í kúluhnitum er þar með fundinn

$$\nabla f = \frac{\partial\varphi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}\vec{e}_\phi. \tag{D.2.4}$$

□

D.3 Sundurleitni í pólhnitum og kúluhnitum

Nú skulum við líta á sundurleitni vigursviðsins $\vec{v} : U \to \mathbb{R}^3$. Hún er sett fram í rétthyrndum hnitum sem

$$\nabla\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \qquad \text{ef} \quad \vec{v} = \vec{v}_x\vec{e}_x + v_y\vec{e}_y + v_z\vec{e}_z.$$

Ef $D\vec{v}(\vec{r})$ táknar afleiðu vigursviðsins í punktinum $\vec{r} = (x, y, z)$, þá er fylki $D\vec{v}(\vec{r})$ miðað við grunninn $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ gefið með formúlunni

$$\begin{bmatrix} \partial_x v_x & \partial_y v_x & \partial_z v_x \\ \partial_x v_y & \partial_y v_y & \partial_z v_y \\ \partial_x v_z & \partial_y v_z & \partial_z v_z \end{bmatrix}.$$

og $\nabla\cdot\vec{v}$ er summa hornalínustakanna í fylkinu. Fyrir $n\times n$ fylki A er summa hornalínustakanna nefnd spor A og er táknuð með trace$A$. Það er auðvelt að sannfæra sig um að $\text{trace}(B^{-1}AB) = \text{trace}A$ fyrir sérhvert andhverfanlegt fylki. Af því leiðir að sundurleitnin $\nabla\cdot\vec{v}$ er óháð því hvaða fylkjaframsetning er tekin á $D\vec{v}(\vec{r})$.

Nú setjum við $\vec{w}(\xi, \eta, \zeta) = \vec{v}(\vec{r}(\xi, \eta, \zeta))$ og skrifum $\vec{w} = v_\xi\vec{e}_\xi + v_\eta\vec{e}_\eta + v_\zeta\vec{e}_\zeta$. Keðjureglan gefur okkur að

$$D\vec{v}(\vec{r})\vec{e}_\xi = \frac{1}{a}\partial_\xi\vec{w}, \qquad D\vec{v}(\vec{r})\vec{e}_\eta = \frac{1}{b}\partial_\eta\vec{w}, \qquad D\vec{v}(\vec{r})\vec{e}_\zeta = \frac{1}{c}\partial_\zeta\vec{w},$$

og þar með er

$$\nabla\cdot\vec{v} = \frac{1}{a}\big(\partial_\xi\vec{w}\big)\cdot\vec{e}_\xi + \frac{1}{b}\big(\partial_\eta\vec{w}\big)\cdot\vec{e}_\eta + \frac{1}{c}\big(\partial_\zeta\vec{w}\big)\cdot\vec{e}_\zeta.$$

Tökum nú fyrir sértilfellið $\vec{w} = \vec{e}_\xi$. Fyrst $\vec{e}_\xi$ er einingarvigur, þá er $0 = \partial_\xi\big(\vec{e}_\xi\cdot\vec{e}_\xi\big) = 2\big(\partial_\xi\vec{e}_\xi\big)\cdot\vec{e}_\xi$ og því

$$\nabla\cdot\vec{v} = \frac{1}{abc}\big[ac\big(\partial_\eta\vec{e}_\xi\big)\cdot\vec{e}_\eta + ab\big(\partial_\zeta\vec{e}_\xi\big)\cdot\vec{e}_\zeta\big]$$

Nú athugum við að

$$\begin{aligned}
a\big(\partial_\eta\vec{e}_\xi\big)\cdot\vec{e}_\eta &= \big(\partial_\eta(a\vec{e}_\xi)\big)\cdot\vec{e}_\eta = \big(\partial_\eta\partial_\xi\vec{r}\big)\cdot\vec{e}_\eta = \big(\partial_\xi\partial_\eta\vec{r}\big)\cdot\vec{e}_\eta \\
&= \big(\partial_\xi(b\vec{e}_\eta)\big)\cdot\vec{e}_\eta = \partial_\xi b
\end{aligned}$$

Í síðasta skrefinu notfærðum við okkur að $\vec{e}_\eta$ er einingarvigur og þar með er $0 = \partial_\xi(\vec{e}_\eta \cdot \vec{e}_\eta) = 2(\partial_\xi \vec{e}_\eta) \cdot \vec{e}_\eta$. Á nákvæmlega sama hátt fáum við síðan að

$$a(\partial_\zeta \vec{e}_\xi) \cdot \vec{e}_\zeta = \partial_\xi c.$$

Við höfum því formúluna

$$\mathrm{div}\vec{v} = \frac{1}{abc}\left[(\partial_\xi b)c + b(\partial_\xi c)\right] = \frac{1}{abc}\partial_\xi(bc).$$

Þetta var sértilfellið $\vec{w} = \vec{e}_\xi$. Lítum nú á $\vec{w} = v_\xi \vec{e}_\xi$ og notfærum okkur formúluna $\nabla \cdot (f\vec{F}) = \nabla f \cdot \vec{F} + f\nabla \cdot \vec{F}$. Hún gefur ásamt D.2.1 að

$$\mathrm{div}\vec{v} = \frac{1}{a}\partial_\xi v_\xi + \frac{1}{abc} v_\xi \partial_\xi(bc) = \frac{1}{abc}\partial_\xi(v_\xi bc).$$

Með því að skipta á hlutverkum breytistærðanna í þessari formúlu fáum við sams konar formúlur fyrir $\nabla \cdot v$ í tilfellunum $\vec{v} = v_\eta \vec{e}_\eta$ og $\vec{v} = v_\zeta \vec{e}_\zeta$, en þær eru $\nabla \cdot (v_\eta \vec{e}_\eta) = \partial_\eta(v_\eta ac)/abc$ og $\nabla \cdot (v_\zeta \vec{e}_\zeta) = \partial_\zeta(v_\zeta ab)/abc$ Almenna formúlan er þar með fundin

(D.3.1) $$\nabla \cdot \vec{v} = \frac{1}{abc}\left[\frac{\partial}{\partial \xi}(bcv_\xi) + \frac{\partial}{\partial \eta}(acv_\eta) + \frac{\partial}{\partial \zeta}(abv_\zeta)\right].$$

Ef vigursviðið $\vec{v}$ er aðeins háð tveimur breytistærðum (x, y) og við veljum $z = \zeta$, þá er $c = 1$ og við fáum sértilfellið

(D.3.2) $$\nabla \cdot \vec{v} = \frac{1}{ab}\left[\frac{\partial}{\partial \xi}(bv_\xi) + \frac{\partial}{\partial \eta}(av_\eta)\right].$$

Sýnidæmi D.3.1 (*Sundurleitni í pólhnitum*). Nú skulum við skrifa $\vec{v} = v_r \vec{e}_r + v_\theta \vec{e}_\theta$ með sama rithætti og í sýnidæmi D.2.1. Þá er $\xi = r$, $\eta = \theta$, $a = 1$ og $b = r$. Þar með er

(D.3.3) $$\nabla \cdot \vec{v} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rv_r) + \frac{\partial}{\partial \theta}v_\theta\right].$$

□

Sýnidæmi D.3.2 Sundurleitni í kúluhnitum Við skrifum $\vec{v} = v_r \vec{e}_r + v_\theta \vec{e}_\theta + v_\phi \vec{e}_\phi$ og stingum $\xi = r$, $\eta = \theta$, $\zeta = \phi$, $a = 1$, $b = r$ og $c = r\sin\theta$ inn í (D.3.2). Þá fáum við

(D.3.4) $$\begin{aligned}\nabla \cdot \vec{v} &= \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta v_r) + \frac{\partial}{\partial \theta}(r\sin\theta v_\theta) + \frac{\partial}{\partial \phi}(rv_\phi)\right] \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}.\end{aligned}$$

□

D.4 Laplace-virki í pólhnitum og kúluhnitum

Nú lítum við aftur á almenn hornrétt hnit og gerum ráð fyrir að $\vec{v}$ sé stigulsvið $\vec{v} = \nabla f$. Við skrifum $f(x, y, z) = \varphi(\xi, \eta, \zeta)$ og höfum því

$$\vec{v} = \frac{1}{a}\frac{\partial \varphi}{\partial \xi}\vec{e}_\xi + \frac{1}{b}\frac{\partial \varphi}{\partial \eta}\vec{e}_\eta + \frac{1}{c}\frac{\partial \varphi}{\partial \zeta}\vec{e}_\zeta.$$

og þar með verður formúlan fyrir Laplace-virkjann

(D.4.1) $$\nabla^2 f = \frac{1}{abc}\left[\frac{\partial}{\partial \xi}\left(\frac{bc}{a}\frac{\partial \varphi}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{ac}{b}\frac{\partial \varphi}{\partial \eta}\right) + \frac{\partial}{\partial \zeta}\left(\frac{ab}{c}\frac{\partial \varphi}{\partial \zeta}\right)\right].$$

Í sértifellinu þegar f er aðeins háð tveimur breytistærðum (x,y) og við veljum $z=\zeta$, þá er $c=1$ og við fáum sértilfellið

$$\nabla^2 f = \frac{1}{ab}\left[\frac{\partial}{\partial\xi}\left(\frac{b}{a}\frac{\partial\varphi}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{a}{b}\frac{\partial\varphi}{\partial\eta}\right)\right]. \tag{D.4.2}$$

Sýnidæmi D.4.1 Laplace-virki í pólhnitum Lítum á fall $f\in C^2(U)$ og innleiðum pólhnit eins og í sýnidæmum D.2.1 og D.3.1. Þá fáum við með því að stinga $a=1$ og $b=r$ inn í (D.4.2) að

$$\nabla^2 f = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)\right] = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}. \tag{D.4.3}$$

□

Sýnidæmi D.4.2 Laplace-virki í kúluhnitum Með sama rithætti og í sýnidæmum D.2.2 og D.3.2 fáum við nú

$$\begin{aligned}\nabla^2 f &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\varphi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(r\sin\theta\frac{\partial\varphi}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\varphi}{\partial\phi}\right)\right]\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}.\end{aligned} \tag{D.4.4}$$

□

D.5 Rót í sívalnings- og kúluhnitum

Nú tökum við fyrir *rót* vigursviðsins $\vec v$, en það er táknað með

$$\mathrm{rot}\vec v,\qquad \mathrm{curl}\vec v\qquad \text{eða}\qquad \nabla\times\vec v. \tag{D.5.1}$$

Í rétthyrndum hnitum er rótið gefið með formúlunni

$$\nabla\times\vec v = \left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\vec e_x+\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\vec e_y+\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\vec e_z. \tag{D.5.2}$$

Þessi formúla er oft skrifuð upp á fylkjaformi

$$\nabla\times\vec v = \begin{vmatrix}\vec e_x & \vec e_y & \vec e_z\\ \partial_x & \partial_y & \partial_z\\ v_x & v_y & v_z\end{vmatrix}. \tag{D.5.3}$$

Nú ætlum við að snúa þessari formúlu yfir í hnitin (ξ,η,ζ). Við skrifum því

$$\vec v = v_\xi\vec e_\xi + v_\eta\vec e_\eta + v_\zeta\vec e_\zeta.$$

Við lítum á $\vec e_\xi$, $\vec e_\eta$ og $\vec e_\zeta$ sem vigra í (x,y,z)-rúmi. Þeir eru snertlar við ξ-, η- og ζ-hnitaferlana í stefnu vaxandi gildis á hnitinu. Við getum líka lítið á þá sem einingarvigra í stefnu stigla af hnitaföllunum $(x,y,x)\mapsto(\xi,\eta,\zeta)$. Þar með er

$$\vec v = av_\xi\nabla\xi + bv_\eta\nabla\eta + cv_\zeta\nabla\zeta,$$

þar sem við lítum á $\nabla\xi$, $\nabla\eta$ og $\nabla\zeta$ sem föll af (x,y,z) og stuðlana av_ξ, bv_η og cv_ζ, sem föll af $(\xi(x,y,z),\eta(x,y,z),\zeta(x,y,z))$. Nú gildir formúlan $\nabla\times(F\nabla G)=\nabla F\times\nabla G$ um öll föll F og G af þremur breytistærðum. Hún gefur

$$\nabla\times\vec v = \frac{1}{a}\nabla\big(av_\xi\big)\times\vec e_\xi + \frac{1}{b}\nabla\big(bv_\eta\big)\times\vec e_\eta + \frac{1}{c}\nabla\big(cv_\zeta\big)\times\vec e_\zeta.$$

Nú er $\vec{e}_\xi \times \vec{e}_\eta = \vec{e}_\zeta$, $\vec{e}_\eta \times \vec{e}_\zeta = \vec{e}_\xi$ og $\vec{e}_\zeta \times \vec{e}_\xi = \vec{e}_\eta$, og með því að nota formúluna fyrir stigulinn í (ξ, η, ζ)-hnitunum, þá fáum við

$$\begin{aligned}\nabla \times \vec{v} = &\frac{1}{a}\Big[-\frac{1}{b}\partial_\eta(av_\xi)\vec{e}_\zeta + \frac{1}{c}\partial_\zeta(av_\xi)\vec{e}_\eta\Big]\\ &+\frac{1}{b}\Big[\frac{1}{a}\partial_\xi(bv_\eta)\vec{e}_\zeta - \frac{1}{c}\partial_\zeta(bv_\eta)\vec{e}_\xi\Big]\\ &+\frac{1}{c}\Big[-\frac{1}{a}\partial_\xi(cv_\zeta)\vec{e}_\eta + \frac{1}{b}\partial_\eta(cv_\zeta)\vec{e}_\xi\Big].\end{aligned}$$

Með því að raða liðunum upp á nýtt fáum við

(D.5.4)
$$\begin{aligned}\nabla \times \vec{v} = &\frac{1}{bc}\Big[\frac{\partial}{\partial\eta}(cv_\zeta) - \frac{\partial}{\partial\zeta}(bv_\eta)\Big]\vec{e}_\xi\\ &+\frac{1}{ac}\Big[\frac{\partial}{\partial\zeta}(av_\xi) - \frac{\partial}{\partial\xi}(cv_\zeta)\Big]\vec{e}_\eta\\ &+\frac{1}{ab}\Big[\frac{\partial}{\partial\xi}(bv_\eta) - \frac{\partial}{\partial\eta}(av_\xi)\Big]\vec{e}_\zeta.\end{aligned}$$

Þeir sem hafa gaman af ákveðum skrifa þessa formúlu sem

(D.5.5)
$$\nabla \times \vec{v} = \frac{1}{abc}\begin{vmatrix} a\vec{e}_\xi & b\vec{e}_\eta & c\vec{e}_\zeta \\ \partial_\xi & \partial_\eta & \partial_\zeta \\ av_\xi & bv_\eta & cv_\zeta \end{vmatrix}.$$

Sýnidæmi D.5.1 Rót í sívalningshnitum Sértilfellið af (D.5.5) fyrir sívalningshnit,

$$(x, y, z) = (r\cos\theta, r\sin\theta, z),$$

er gefið með

(D.5.6)
$$\nabla \times \vec{v} = \frac{1}{r}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & \vec{e}_z \\ \partial_r & \partial_\theta & \partial_z \\ v_r & rv_\theta & v_z \end{vmatrix}.$$

□

Sýnidæmi D.5.2 Rót í kúluhnitum

(D.5.7)
$$\nabla \times \vec{v} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta\vec{e}_\phi \\ \partial_r & \partial_\theta & \partial_\phi \\ v_r & rv_\theta & r\sin\theta v_\phi \end{vmatrix}.$$

□

Lítum nú aftur á vigurssvið $\vec{v} = v_x\vec{e}_x + v_y\vec{e}_y$ í tveimur víddum. Þá er $\vec{w}(x, y, z) = (v_x(x, y), v_y(x, y), 0)$ vigursvið sem er óháð hnitinu z. Þar með er

(D.5.8)
$$\nabla \times \vec{w} = (0, 0, \partial_x v_y - \partial_y v_x).$$

Ef við látum $\vec{v} = v_r\vec{e}_r + v_\theta\vec{e}_\theta$ vera framsetningu á vigursviðinu u í pólhnitum, þá fáum við með því að líta á formúluna fyrir rótið í sívalningshnitum að

(D.5.9)
$$\partial_x v_y - \partial_y v_x = \frac{1}{r}\big(\partial_r(rv_\theta) - \partial_\theta v_r\big).$$

SVÖR VIÐ ÆFINGARDÆMUM

1.1 Skilgreining á nokkrum hugtökum

1.1.1. Nei. Föllin $u(t) = kt$, $k \in \mathbb{R}$ eru öll lausnir.

1.1.3. a) $u' = 2u/x$, b) $u' = x/(1-u)$,
c) $u'(x) = -1/2x$, d) $u' = (u-x)/(x+u)$.

1.2 Fyrsta stigs jöfnur

1.2.1. a) $u(x) = Ce^{3x}/x$ b) $u(x) = x^3(C + \ln|x|)$,
c) $u(x) = (C + \sin x)/(1+x)$, d) $u(x) = e^{x^2}(C + x^3)$.

1.2.2. a) $1/3u^3 - 2/u = C + 1/x + \ln|x|$, b) $\ln|1+u| = C + x + \frac{1}{2}x^2$,
c) $u(x) = \frac{1}{2} + \frac{1}{2}e^{2(x+C)}$.

1.2.3. $u(t) = \arcsin\left(\frac{1}{2}e^{t^2/2}\right)$, $\{t \in \mathbb{R}; |t| \leq \sqrt{\ln 4}\}$.

1.2.5. a) $u(t)^2 = t^2 + Ct$, b) $u(t) = t(\ln|t| + C)$,
c) $u(t) = t(1 - 2/(\ln|t| + C))$, d) $u(t) = -t/(\ln|t| + C)$.

1.2.7. a) $u(t)^{-2} = e^{-2t}\ln(t) + C_1e^{-2t}$, b) $u(t)^3 = e^{-t}t + C_1e^{-t}$,
c) $u(x) = \left(Ce^x - 2x - 1\right)^{-\frac{1}{3}}$, d) $u(x)^2 = 1 + Ce^{-x^2}$.

1.2.8. $u(t) = \exp\left(t^2 + Ct^{-2}\right)$

1.2.10. a) $u(x) = Ce^{x^2} + e^{x^2}\int_0^x e^{-\xi^2}\,d\xi$,
b) $u(x) = x + (c - x)^{-1}$,
c) $u(x) = x^2(e^{x^2} - C)/(e^{x^2} + C)$,
d) $u(x) = x + 5x/(C - x^5)$.

1.2.12. $u(x) = Cx - C^2/4$.

1.5 Jaðargildisverkefni

1.5.2. a) $u(x) = T_0 + \dfrac{3\lambda L^2 + \kappa L^3 - 6\kappa^2 T_0}{6\kappa(\lambda + \kappa L)}x - \dfrac{x^3}{6\kappa}$.

b) $u(x) = T_0 - \dfrac{\left(-\lambda\, L^3 + 3\, v_0\, \kappa\right)}{3\,\kappa\,\lambda}x - \dfrac{x^4}{12\,\kappa}$.

c) $u(x) = T_0 + \dfrac{5\lambda L^4 + \kappa L^5 - 20\kappa^2 T_0}{20\kappa(\lambda + \kappa L)}x - \dfrac{x^5}{20\kappa}$

1.6 Eigingildisverkefni

1.6.1. a) Eigingildin eru $\lambda_n = (n\pi/L)^2$ og tilsvarandi eiginföll $u_n(x) = C_n \cos(n\pi x/L)$, $C_n \neq 0$, $n = 0, 1, 2,.$
b) Eigingildin eru $\lambda_n = ((n+\frac{1}{2})\pi/L)^2$ og tilsvarandi eiginföll $u_n(x) = C_n \sin((n+\frac{1}{2})\pi x/L)$, $C_n \neq 0$, $n = 0, 1, 2,.$

1.6.2. a) $u(x,t) = (A_n \cos(n\pi ct/L) + B_n \sin(n\pi ct/L)) \cos(n\pi x/L)$,
b) $u(x,t) = A_n \exp(-\kappa((n+\frac{1}{2})\pi/L)^2 t) \sin((n+\frac{1}{2})\pi x/L)$, $n = 0, 1, 2, \dots$.

1.7 Tilvist og ótvíræðni lausna

1.7.1. $u_0(t) = 0$, $u_1(t) = t$, $u_2(t) = t + \frac{1}{3}t^2$, $u_3(t) = t + \frac{1}{3}t^2 + \frac{2}{15}t^5 + \frac{1}{63}t^7$, $u(t) = \tan t = t + \frac{1}{3}t^2 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \cdots$.

2.1 Línulegir afleiðuvirkjar

2.1.1. a) $P(t,D)Q(t,D) = D^3 + tD^2 + t^2D + (2t + t^3)$,
$Q(t,D)P(t,D) = D^3 + tD^2 + (t^2+2)D + t^3$.
b) $P(t,D)Q(t,D) = tD^2$, $Q(t,D)P(t,D) = tD^2 + D$
c) $P(t,D)Q(t,D) = t^nD^n$, $Q(t,D)P(t,D) = \sum_{k=0}^{n} \binom{n}{k}^2 k! t^{n-k} D^{n-k}$.
d) $P(t,D)Q(t,D) = D^3 - 2D + (-t^3 - t)$, $Q(t,D)P(t,D) = D^3 + D + (-t^3 + 2t)$.

2.1.3. a) háð, b) óháð, c) óháð, d) háð, e) háð, f) háð, g) óháð, h) óháð.

2.1.7. a) $u_2(t) = t^3 \ln t$, b) $u_2(t) = te^t$, c) $u_2(t) = t^{-1}\sin 3t$
d) $u_2(t) = t^{-1}\sinh 2t$, e) $u_2(t) = t^{-1}\sin(t/2)$, f) $u_2(t) = \sqrt{t}\sinh t$.

2.1.8. $\omega \neq 0$.

2.1.9. $L \neq \pi, 2\pi, 3\pi, \dots$.

2.2 Línulegar jöfnur með fastastuðla

2.2.1. a) $u(t) = c_1e^t + c_2e^{3t}$,
b) $u(t) = c_1e^{-2t} + c_2te^{-2t}$,
c) $u(t) = c_1e^{\frac{3}{2}t}\cos(\frac{1}{2}\sqrt{3}\,t) + c_2e^{\frac{3}{2}t}\sin(\frac{1}{2}\sqrt{3}\,t)$,
d) $u(t) = c_1e^{-it} + c_2e^{3it}$,
e) $u(t) = c_1\exp((1+i\sqrt{3})t) + c_2\exp(-(1+i\sqrt{3})t)$,
f) $u(t) = c_1e^t + c_2e^{-t} + c_3te^{-t}$,
g) $u(t) = c_1e^t + c_2e^{-t}\cos 2t + c_3e^{-t}\sin 2t$,
h) $u(t) = c_1\cos t + c_2\sin t + c_3e^{2t} + c_4te^{2t}$,
i) $u(t) = c_1\cos t + c_2\sin t + c_3e^t + c_4e^{2t}$,
j) $u(t) = c_1e^t + c_2e^{\omega t} + c_3e^{\omega^2 t} + c_4e^{\omega^3 t} + c_5e^{\omega^4 t}$, $\omega = e^{i2\pi/5}$,
k) $u(t) = c_1e^{\frac{1}{2}(-3+i\sqrt{3})t} + c_2e^{\frac{1}{2}(-3-i\sqrt{3})t} + c_3te^{\frac{1}{2}(-3+i\sqrt{3})t} + c_4te^{\frac{1}{2}(-3-i\sqrt{3})t}$,
l) $u(t) = c_1e^{-t} + c_2te^{-t} + c_3e^t\cos t + c_4e^t\sin t$.

2.2.2. a) $u(t) = e^{3t}$,
b) $u(t) = 3e^{-2t} + te^{-2t}$,
c) $u(t) = -e^{\frac{3}{2}(t-1)}\cos(\frac{1}{2}\sqrt{3}\,(t-1)) + e^{\frac{3}{2}(t-1)}\sin(\frac{1}{2}\sqrt{3}\,(t-1))$,
d) $u(t) = \left(\frac{21}{4} + \frac{13}{4}i\right)e^{-it} + \left(\frac{7}{4} - \frac{13}{4}i\right)e^{3it}$,

e) $u(t) = 7\exp((1+i\sqrt{3})t) - 2\exp(-(1+i\sqrt{3})t)$,
f) $u(t) = e^t + 2e^{-t} + 4te^{-t}$,
g) $u(t) = \frac{1}{4}\big(3e^{(t-1)} + e^{-(t-1)}\cos 2(t-1) - e^{-(t-1)}\sin 2(t-1)\big)$,
h) $u(t) = \cos t - \sin t + e^{2t} - te^{2t}$,
i) $u(t) = 2\cos t + 3\sin t + e^t + e^{2t}$,
j) $u(t) = e^t + \omega e^{\omega t} + \omega^2 e^{\omega^2 t} + \omega^3 e^{\omega^3 t} + \omega^4 e^{\omega^4 t}$, $\omega = e^{i2\pi/5}$,
k) $u(t) = e^{-\frac{3}{2}t}(\cos(\frac{\sqrt{3}}{2}t) + \frac{169\sqrt{3}}{9}\sin(\frac{\sqrt{3}}{2}t) - \frac{74}{3}t\cos(\frac{\sqrt{3}}{2}t) + 4\sqrt{3}t\sin(\frac{\sqrt{3}}{2}t))$.
l) $u(t) = \frac{-11}{25}e^{-(t-1)} + \frac{1}{5}(t-1)e^{-(t-1)} + \frac{36}{25}e^{(t-1)}\cos(t-1) - \frac{2}{25}e^{(t-1)}\sin(t-1)$.

2.2.3. a) $u(t) = c_1e^t + c_2e^{3t} + (2/10)\cos t + (1/10)\sin t$,
$c_1 = \big(-(2/10)e^{2L} + (2/10)\cos L + (1/10)\sin L\big)e^{-L}(e^L-1)^{-1}$
$c_2 = \big((2/10)e^L - (2/10)\cos L - (1/10)\sin L\big)e^{-L}(e^L-1)^{-1}$
b) $u(t) = c_1e^{-2t} + c_2te^{-2t} - (4/25)\cos t + (3/25)\sin t$
$c_1 = 4/25$, $c_2 = ((4/25)\cos L - (3/25)\sin L - (4/25)e^{-2L})/(Le^{-2L})$
f) $u(t) = c_1e^t + c_2e^{-t} + c_3te^{-t} - (1/4)\cos t - (1/4)\sin t$
$c_1 = 1/4 - e^L(\cos L + \sin L)/(16L)$,
$c_2 = e^L(\cos L + \sin L)/(16L)$,
$c_3 = e^L(\cos L + \sin L)/(8L)$.
g) $u(t) = c_1e^t + c_2e^{-t}\cos 2t + c_3e^{-t}\sin 2t - (3/20)\cos t + (1/20)\sin t$,
$c_2 = \big(e^L(3\cos L - \sin L)/80 + (\sin 2L - \cos 2L)/20\big)/\sin 2L$,
$c_1 = 3/20 - c_2$, $c_3 = -1/10 + c_2$.

2.3 Euler-jöfnur

2.3.2. a) $u(x) = c_1x^{-3} + c_2x^2$,
b) $u(x) = x^{-3}\big(c_1\cos(2\ln x) + c_2\sin(2\ln x)\big)$,
c) $u(x) = c_1x^{-\frac{1}{2}} + c_2x^3$,
d) $u(x) = x\big(c_1\cos(\ln x) + c_2\sin(x)\big)$,
e) $u(x) = x^2\big(c_1 + c_2\ln x\big)$,
f) $u(x) = c_1x^{-2} + c_2x^2 + c_3$,
g) $u(x) = x^{-1}\big(c_1 + c_2\ln x + c_3(\ln x)^2\big)$,
h) $u(x) = c_1x + c_2x^3 + c_3x^{1+i} + c_4x^{1-i}$.

2.4 Sérlausnir

2.4.1. $u_p(t) = -\dfrac{3}{25}\cos 2t - \dfrac{4}{25}\sin 2t$.

2.4.2. a) $u_p(t) = (2/25)t\sin t - (3/50)t\cos t$,
b) $u_p(t) = (1/4)e^t - (1/36)e^{-t}$,
c) $u_p(t) = (t^2/20)e^{2t} - (1/160)e^{-2t}$,
d) $u_p(t) = -(1/845)e^{-t}\cos 2t - (29/1690)e^{-t}\sin 2t$.

2.5 Green–föll

2.5.2. a) $G(t,\tau) = -\frac{1}{2}e^{(t-\tau)} + \frac{1}{2}e^{3(t-\tau)}$,
b) $G(t,\tau) = (t-\tau)e^{-2(t-\tau)}$,
c) $G(t,\tau) = \dfrac{2}{\sqrt{3}}e^{\frac{3}{2}(t-\tau)}\sin(\sqrt{3}(t-\tau)/2)$,

d) $G(t,\tau) = \dfrac{i}{4}e^{-i(t-\tau)} - \dfrac{i}{4}e^{3i(t-\tau)}$,
e) $G(t,\tau) = \dfrac{e^{(1+i\sqrt{3})(t-\tau)}}{2(1+i\sqrt{3})} - \dfrac{e^{-(1+i\sqrt{3})(t-\tau)}}{2(1+i\sqrt{3})}$,
f) $G(t,\tau) = \dfrac{1}{4}e^{(t-\tau)} - \dfrac{1}{4}e^{-(t-\tau)} - \dfrac{1}{2}(t-\tau)e^{-(t-\tau)}$,
g) $G(t,\tau) = \big(e^{(t-\tau)} - e^{-(t-\tau)}\big(\cos 2(t-\tau) + \sin 2(t-\tau)\big)\big)/8$,
h) $G(t,\tau) = \big(3\sin(t-\tau) + 4\cos(t-\tau) - 4e^{2(t-\tau)} + 5(t-\tau)e^{2(t-\tau)}\big)/25$,
i) $G(t,\tau) = \big(3\cos(t-\tau) + \sin(t-\tau) - 5e^{(t-\tau)} + 2e^{2(t-\tau)}\big)/10$,
j) $G(t,\tau) = \dfrac{1}{5}\sum_{j=0}^{4}\omega^j e^{\omega^j(t-\tau)}$,
k) $G(t,\tau) = \dfrac{4}{9}\sqrt{3}e^{-3(t-\tau)/2}\sin\big(\sqrt{3}(t-\tau)/2\big) - \dfrac{2}{3}(t-\tau)e^{-3(t-\tau)/2}\cos\big(\sqrt{3}(t-\tau)/2\big)$.
l) $G(t,\tau) = \big(e^{t-\tau}\big(3\sin(t-\tau) - 4\cos(t-\tau)\big)\big) + e^{-(t-\tau)}(4 + 5(t-\tau))\big)/25$.

2.6 Wronski–fylkið og Wronski–ákveðan

2.6.1. a) $G(t,\tau) = -(\ln\tau)t^3/\tau^4 + t^3(\ln t)/\tau^4$,
b) $G(t,\tau) = -t/\tau^3 + te^{t-\tau}/\tau^3$,
c) $G(t,\tau) = (1/3t)\sin 3(t-\tau)$,
d) $G(t,\tau) = (1/2t)\sinh 2(t-\tau)$,
e) $G(t,\tau) = (1/2t)\sin\frac{1}{2}(t-\tau)$,
f) $G(t,\tau) = (1/\tau^3)\sqrt{t/\tau}\sinh(t-\tau)$,

2.7 Green-föll fyrir jaðargildisverkefni

2.7.1. a) $G_B(x,\xi) = \begin{cases} \xi, & 0 \le \xi \le x \le 1, \\ x, & 0 \le x \le \xi \le 1, \end{cases}$

b) $G_B(x,\xi) = \begin{cases} \xi(1 - hx/(1+h)), & 0 \le \xi \le x \le 1, \\ x(1 - h\xi/(1+h)), & 0 \le x \le \xi \le 1, \end{cases}$

c) $G_B(x,\xi) = \begin{cases} \dfrac{\sin(\omega\xi)\cos(\omega(x-1))}{\omega\cos\omega}, & 0 \le \xi \le x \le 1, \\ \dfrac{\sin(\omega x)\cos(\omega(\xi-1))}{\omega\cos\omega}, & 0 \le x \le \xi \le 1, \end{cases}$

d) $G_B(x,\xi) = \begin{cases} \dfrac{\sinh(\omega\xi)\sinh(\omega(1-x))}{\omega\sinh\omega}, & 0 \le \xi \le x \le 1, \\ \dfrac{\sinh(\omega x)\sinh(\omega(1-\xi))}{\omega\sinh\omega}, & 0 \le x \le \xi \le 1, \end{cases}$

e) $G_B(x,\xi) = \begin{cases} \dfrac{\sinh(\omega\xi)(\omega\cosh(\omega(x-1)) - h\sinh(\omega(x-1)))}{\omega^2\cosh\omega + h\omega\sinh\omega}, & 0 \le \xi \le x \le 1, \\ \dfrac{\sinh(\omega x)(\omega\cosh(\omega(\xi-1)) - h\sinh(\omega(\xi-1)))}{\omega^2\cosh\omega + h\omega\sinh\omega}, & 0 \le x \le \xi \le 1. \end{cases}$

2.7.2. $G_B(x,\xi) = \begin{cases} \frac{1}{2}x^2 + \frac{1}{2}\xi^2 - x, & 0 \le \xi \le x \le 1, \\ \xi x - x, & 0 \le x \le \xi \le 1. \end{cases}$

2.7.3. $G_B(x,\xi) = \begin{cases} \dfrac{x}{\xi}(x-2)(\xi-1), & 1 \le \xi \le x \le 2, \\ \dfrac{x}{\xi}(\xi-2)(x-1), & 1 \le x \le \xi \le 2. \end{cases}$

3.2 Raðalausnir umhverfis venjulega punkta

3.2.1. Hér notum við ritháttinn $(2n+1)!! = 1\cdot 3\cdot 5\cdots(2n+1)$.
a) $c_{n+2} = c_n$, $\varrho = 1$,
$u(x) = c_0\sum_{n=0}^\infty x^{2n} + c_1\sum_{n=0}^\infty x^{2n+1} = (c_0+c_1x)/(1-x^2)$.
b) $c_{n+2} = -c_n/(n+2)$, $\varrho = \infty$,
$u(x) = c_0\sum_{n=0}^\infty[(-1)^n/n!\cdot 2^n]x^{2n} + c_1\sum_{n=0}^\infty[(-1)^n/(2n+1)!!]x^{2n+1}$,
c) $c_{n+2} = [(n-4)(n+4)/2(n+1)(n+2)]c_n$, $\varrho = \sqrt{2}$,
$u(x) = c_0(1-4x^2+2x^4)$
$+c_1(x-\frac{5}{4}x^3+\frac{7}{32}x^5+\sum_{n=3}^\infty[(2n-5)!!(2n+3)!!/(2n+1)!2^n]x^{2n+1}$,
d) $c_2 = 0$, $c_{n+3} = -c_n/[(n+2)(n+3)]$, $\varrho = +\infty$,
$u(x) = c_0\big(1+\sum_{n=1}^\infty(-1)^n x^{3n}/[3^n n!\cdot 2\cdot 5\cdots(3n-1)]\big)$
$+c_1\sum_{n=0}^\infty(-1)^n x^{3n+1}/[3^n\cdot n!\cdot 1\cdot 4\cdots(3n+1)]$,
e) $c_2 = c_3 = 0$, $c_{n+4} = -c_n/[(n+3)(n+4)]$, $\varrho = +\infty$,
$u(x) = c_0\big(1+\sum_{n=1}^\infty(-1)^n x^{4n}/[4^n\cdot n!\cdot 3\cdot 7\cdots(4n-1)]\big)$
$+c_1\sum_{n=0}^\infty(-1)^n x^{4n+1}/[4^n\cdot n!\cdot 5\cdot 9\cdots(4n+1)]$.

3.2.2. a) $2c_2 + c_0 = 0$, $(n+2)(n+1)c_{n+2} + c_n + c_{n-1} = 0$,$n \ge 1$,
$u_1(x) = 1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots$, $u_2(x) = x - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \cdots$.
b) $(n+2)(n+1)c_{n+2} = (n^2+n)c_n + 2c_{n-1}$
$u_1(x) = 1 + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots$, $u_2(x) = x + \frac{1}{3}x^3 + \frac{1}{6}x^4 + \cdots$,
c) $c_2 = c_3 = 0$, $(n+3)(n+4)c_{n+4} + (n+1)c_{n+1} + c_n = 0$, $n \ge 0$,
$u_1(x) = 1 - \frac{1}{12}x^4 + \frac{1}{126}x^7 + \cdots$, $u_2(x) = x - \frac{1}{12}x^4 - \frac{1}{20}x^5 + \cdots$.
d) $c_2 = c_3 = c_4 = c_5 = 0$, $(n+2)(n+1)c_{n+2} + (n-1)(n-2)c_{n-1} + c_{n-4} = 0$, $n \ge 0$.
$u_1(x) = 1 - \frac{1}{30}x^6 + \frac{1}{72}x^9 + \cdots$, $u_2(x) = x - \frac{1}{42}x^7 + \frac{1}{90}x^{10} + \cdots$.

3.4 Reglulegir sérstöðupunktar – Aðferð Frobeniusar

3.4.1. Lausnin er línuleg samantekt fallanna u_1 og u_2 sem gefin eru með formúlunum:
a) $u_1(x) = x^{1/3}\left(1+\sum_{n=1}^\infty \dfrac{(-1)^n x^{2n}}{2^n\cdot n!\cdot 7\cdot 13\cdots(6n+1)}\right)$,
$u_2(x) = 1+\sum_{n=1}^\infty \dfrac{(-1)^n x^{2n}}{2^n\cdot n!\cdot 5\cdot 11\cdots(6n-1)}$.
b)$u_1(x) = x^{-1}\cos 2x$, $u_2(x) = x^{-1}\sin 2x$,
c) $u_1(x) = e^x$, $\quad u_2(x) = e^x\ln x$,
d) $u_1(x) = x^3$, $\quad u_2(x) = x^3\ln x$,
e) $u_1(x) = \sum_{n=0}^\infty \dfrac{x^n}{n!(2n+1)!!}$, $\quad u_2(x) = \sqrt{x}\sum_{n=0}^\infty \dfrac{x^n}{n!(2n+1)!!}$,
f) $u_1(x) = x^{3/2}\left(1+\sum_{n=1}^\infty \dfrac{(-1)^n x^{2n}}{n!\cdot 5\cdot 9\cdot 13\cdots(4n+1)}\right)$,
$u_2(x) = x^{-1}\left(\sum_{n=1}^\infty \dfrac{(-1)^{n-1}x^{2n}}{n!\cdot 3\cdot 7\cdots(4n-1)}\right)$),
g) $u_1(x) = x^{\frac{1}{2}}e^{-x/2}$, $\quad u_2(x) = 1+\sum_{n=1}^\infty \dfrac{(-1)^n x^n}{(2n-1)!!}$,
h) $u_1(x) = \sum_{n=0}^\infty x^{2k}/2^{2k}(k!)^2$, $\quad u_2(x) = u_1(x)\left(\ln x - \dfrac{x^2}{4} + \dfrac{5x^4}{128} - \dfrac{23x^6}{3456}\right)$.

3.4.2. a) venjulegur punktur, b) venjulegur punktur, c) óreglulegur sérstöðupunktur, d) óreglulegur sérstöðupunktur, e) reglulegur sérstöðupunktur, $r_1 = 0$, $r_2 = -1$, f) reglulegur sérstöðupunktur, $r_1 = 1$, $r_2 = -2$, g) reglulegur sérstöðupunktur, $r_1 = -2$, $r_2 = -3$, h) reglulegur sérstöðupunktur, $r_1 = \frac{1}{2}$, $r_2 = -3$.

3.5 Bessel-jafnan

3.5.8. a) $u(x) = x^{\frac{1}{2}}\left[c_1 J_{1/3}(\frac{2}{3}x^{\frac{3}{2}}) + c_2 J_{-1/3}(\frac{2}{3}x^{\frac{3}{2}})\right]$,
b) $u(x) = x^{\frac{1}{2}}\left[c_1 J_{1/4}(\frac{1}{2}x^2) + c_2 J_{-1/4}(\frac{1}{2}x^2)\right]$,
c) $u(x) = x^{\frac{1}{2}}\left[c_1 J_\alpha(2\alpha x^{1+m/2}) + c_2 J_{-\alpha}(2\alpha x^{1+m/2})\right]$, $\alpha = 1/(m+2)$,
d) $u(x) = x^{\frac{1}{2}}\left[c_1 J_\alpha(\frac{1}{2}x^2) + c_2 J_{-\alpha}(\frac{1}{2}x^2)\right]$, $\alpha = \sqrt{2}/8$.

4.1 Varpanir á tvinntöluplaninu

4.1.1. a) Línan sem er hornrétt á strikið milli punktanna 1 og -2 og inniheldur punktinn mitt á milli þeirra. Jafna hennar er $x = -1/2$.
b) Línan sem er hornrétt á strikið milli punktanna $1+i$ og $-1-i$ og inniheldur punktinn mitt á milli þeirra. Jafna hennar er $y = -x$.
c) Hringur með geisla $1/\sqrt{\alpha - 1}$ og miðju 0.
d) Hringur með geisla $1/\sqrt{1 - \alpha}$ og miðju 0.

4.1.2. $|A| \neq |B|$.

4.2 Fáguð föll

4.2.1. a) f er ekki fágað, en $f'(z)$ er til í $z = 0$ og $f'(0) = 0$.
b) f er fágað á $\{z; z \neq 2\}$ og $f'(z) = -1/(z-2)^2$.
c) f er fágað á $\{z; z \neq 0\}$ og $f'(z) = 1 - 1/z^2$.
d) f er fágað á $\{z; z \neq \pm 1\}$ og $f'(z) = -2z/(z^2-1)^2$.
e) f er ekki fágað, en $\mathbb{C}$-afleiðan er til í $z = 0$ og $f'(0) = 0$.
f) f er ekki fágað.
g) $f(z) = \bar{z}$ er ekki fágað.
h) $f(z) = 1/z$ er fágað á $\{z; z \neq 0\}$ og $f'(z) = -1/z^2$.

4.2.2. a) $f(z) = z^3 + z - 4$ er fágað,
b) $f(z) = \bar{z}^2$ er ekki fágað,
c) f er fágað á menginu $\{z = x + iy; x \neq 0\}$.
d) f er fágað á menginu $\{z = x + iy; y \neq 0\}$.

4.4 Stofnbrotaliðun

4.4.1. a) $-\dfrac{1}{2z} - \dfrac{3}{2(z+2)} + \dfrac{2}{z-1}$,
b) $\dfrac{-7/5 + i9/5}{z+i} + \dfrac{-i}{z-i} + \dfrac{7/5 - 4i/5}{z+2}$,
c) $\dfrac{2}{4z+1} + \dfrac{1}{z+3}$,
d) $\dfrac{z^2}{(z^2+1)^2}$,
e) $\dfrac{3}{4(z-1)} - \dfrac{3}{4(z+1)} + \dfrac{3}{2(z+1)^2}$,
f) $\dfrac{1}{z} - \dfrac{1}{z-1} + \dfrac{1}{(z-1)^2}$,
g) $2z + \dfrac{1}{2(z-1)} + \dfrac{3}{2(z+1)} - \dfrac{1}{3(z+1)^2} + \dfrac{2\sqrt{3}i}{9(z - 1/2 - i\sqrt{3}/2)} - \dfrac{2\sqrt{3}i}{9(z - 1/2 + i\sqrt{3}/2)}$.

4.4.3. $a_n = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right].$

4.4.4. $f(z) = P(z)/Q(z)$, þar sem

$$\begin{aligned} P(z) &= a_0 + (a_1 - \alpha_1 a_0)z + (a_2 - \alpha_1 a_1 - \alpha_2 a_0)z^2 \\ &\quad + \cdots + (a_{m-1} - \alpha_1 a_{m-2} - \cdots - \alpha_{m-1} a_0)z^{m-1} \\ Q(z) &= 1 - \alpha_1 z - \alpha_2 z^2 - \cdots - \alpha_m z^m. \end{aligned}$$

4.5 Veldisvísisfallið og skyld föll

4.5.2. a) lota: 2π, núllstöðvar: $\pi/2 + n\pi$, $n \in \mathbb{Z}$, afleiða: $-\sin z$,
b) lota: 2π, núllstöðvar: $n\pi$, $n \in \mathbb{Z}$, afleiða: $\cos z$,
c) lota: π, núllstöðvar: $n\pi$, $n \in \mathbb{Z}$ afleiða: $1/\cos^2 z$,
d) lota: $2\pi i$, núllstöðvar: $\pi i/2 + n\pi i$, $n \in \mathbb{Z}$, afleiða: $\sinh z$,
e) lota: $2\pi i$, núllstöðvar: $n\pi i$, $n \in \mathbb{Z}$, afleiða: $\cosh z$,
f) lota: $i\pi$, núllstöðvar: $n\pi i$, afleiða: $1/\cosh^2 z$.

4.6 Lograr, rætur og horn

4.6.4. a) $-i\pi/2$, b) $(\ln 2)/2 + i\pi/4$, c) $4\ln 2 + i4\pi/3$, d) $4\ln 2 - i2\pi/3$,
e) $-\pi^2/4 + i\pi(\ln 2)/2$, f) $e^{\pi/2}$, g) $\cos\big(\pi \ln 2\big) + i \sin\big(\pi \ln 2\big)$
h) $\sqrt{2}e^{-\pi/4}\big(\cos\big(\frac{1}{2}\ln 2 + \frac{1}{4}\pi\big) + i\sin\big(\frac{1}{2}\ln 2 + \frac{1}{4}\pi\big)\big)$, i) 1.

4.6.5. a) $e^{\pi/4}$, b) $i\pi/2 + 2\pi i n$, $n \in \mathbb{Z}$,
c) $-i\ln(\sqrt{2}+1) + (2n+1)\pi$, $-i\ln(\sqrt{2}-1) + 2n\pi$, $n \in \mathbb{Z}$,
d) engin lausn.

5.2 Hneppi með fastastuðla

5.2.1. a)$\lambda_1 = 3 - 2\sqrt{2}$, $\lambda_2 = 3 + 2\sqrt{2}$, $\varepsilon_1 = (-1, 1+\sqrt{2})$, $\varepsilon_2 = (-1, 1-\sqrt{2})$.
b) $\lambda_1 = 1$, $\lambda_2 = -9$, $\varepsilon_1 = (1,1)$, $\varepsilon_2 = (2,-3)$.
c) $\lambda_1 = 3+4i$, $\lambda_2 = 3-4i$, $\varepsilon_1 = (1,-i)$, $\varepsilon_2 = (1,i)$.
d) $\lambda_1 = 4i$, $\lambda_2 = -4i$, $\varepsilon_1 = (5, 2-4i)$, $\varepsilon_2 = (5, 2+4i)$.

5.2.2. a) $u(t) = \frac{1-\sqrt{2}}{2\sqrt{2}} e^{(3-2\sqrt{2})t}\begin{bmatrix} -1 \\ 1+\sqrt{2}\end{bmatrix} - \frac{1+\sqrt{2}}{2\sqrt{2}} e^{(3+2\sqrt{2})t}\begin{bmatrix} -1 \\ 1-\sqrt{2}\end{bmatrix}$,
b) $u(t) = \frac{7}{5}e^t\begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{5}e^{-9t}\begin{bmatrix}2\\-3\end{bmatrix}$,
c) $u(t) = (1+\frac{1}{2}i)e^{(3+4i)t}\begin{bmatrix}1\\-i\end{bmatrix} + (1-\frac{1}{2}i)e^{(3-4i)t}\begin{bmatrix}1\\i\end{bmatrix} = e^{3t}\begin{bmatrix}2\cos 4t - \sin 4t\\ \cos 4t + 2\sin 4t\end{bmatrix}$,
d) $u(t) = -\frac{1}{8i}e^{4it}\begin{bmatrix}5\\2-4i\end{bmatrix} + \frac{1}{8i}e^{-4it}\begin{bmatrix}5\\2+4i\end{bmatrix} = \begin{bmatrix}-\frac{5}{4}\sin 4t\\ \cos 4t - \frac{1}{2}\sin 4t\end{bmatrix}$.

5.2.3. a) $u_p(t) = -\dfrac{1}{12}e^t\begin{bmatrix}3\\2\end{bmatrix}$,
b) $u_p(t) = \dfrac{1}{11}e^{2t}\begin{bmatrix}4\\5\end{bmatrix}$,

c) $u_p(t) = \dfrac{1}{102}\begin{bmatrix}\cos t - 13\sin t\\ 4\cos t + 16\sin t\end{bmatrix}$,
d) $u_p(t) = -\dfrac{1}{15}\begin{bmatrix}5\cos t\\ 2\cos t + \sin t\end{bmatrix}$.

5.2.5. a) $\lambda_1 = 0$, $\lambda_2 = 3$, $\lambda_3 = -1$, $\varepsilon_1 = (1, -1, 1)$, $\varepsilon_2 = (1, 2, 1)$, $\varepsilon_3 = (1, 0, -1)$.
b) $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\varepsilon_1 = (-1, 1, 2)$, $\varepsilon_2 = (-2, 1, 4)$, $\varepsilon_3 = (-1, 1, 4)$.

5.2.6. a) $u(t) = \frac{2}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix} + \frac{4}{3}e^{3t}\begin{bmatrix}1\\2\\1\end{bmatrix} - e^{-t}\begin{bmatrix}1\\0\\-1\end{bmatrix}$,
b) $u(t) = e^{2t}\begin{bmatrix}-2\\1\\4\end{bmatrix} - e^{3t}\begin{bmatrix}-1\\1\\4\end{bmatrix}$.

5.5 Veldisvísisfylkið

5.5.3. (c) $u(t) = \begin{bmatrix}-e^t + 2te^t + 1\\ 2te^t\end{bmatrix}$

5.7 Newton–margliður

5.7.1. a) $e^{tA} = e^{(3-2\sqrt{2})t}I + \frac{1}{4\sqrt{2}}(e^{(3+2\sqrt{2})t} - e^{(3-2\sqrt{2})t})(A - (3 - 2\sqrt{2})I)$,
b) $e^{tA} = e^tI - \frac{1}{10}(e^{-9t} - e^t)(A - I)$,
c) $e^{tA} = (e^{3t}\cos 4t)I + (\frac{1}{4}e^{3t}\sin 4t)(A - 3I)$,
d) $e^{tA} = (\cos 4t)I + (\frac{1}{4}\sin 4t)A$.

5.7.2. a) $e^{tA} = I - \frac{1}{3}(1 - e^{3t})A + (-\frac{1}{3} + \frac{1}{12}e^{3t} + \frac{1}{4}e^{-t})A(A - 3I)$.
b) $e^{tA} = e^tI + (e^{2t} - e^t)(A - I) + \frac{1}{2}(e^{3t} - 2e^{2t} - e^t)(A - I)(A - 2I)$.

5.7.3. $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 2$, $\varepsilon_1 = (-1, -1, 1)$, $\varepsilon_2 = (0, 1, -1)$.

$$e^{tA} = e^tI + (e^{2t} - e^t)(A - I) + (e^{2t} + e^t)(A - I)(A - 2I),$$
$$u(t) = 6e^t\begin{bmatrix}1\\-1\\1\end{bmatrix} - e^{2t}\begin{bmatrix}5\\-8\\3\end{bmatrix}.$$

5.7.4. $e^{tA} = e^{2t}I + te^{2t}(A - 2I) + \frac{1}{2}t^2e^{2t}(A - 2I)^2$.
$u(t) = e^{2t}\begin{bmatrix}1\\0\\1\end{bmatrix} + te^{2t}\begin{bmatrix}1\\-1\\1\end{bmatrix}$.

5.7.5.

$$
\begin{aligned}
e^{tA} &= e^t I + te^t(A-I) + \big(\frac{t}{2}e^t - \frac{1}{4}e^t + \frac{1}{4}e^{-t}\big)(A-I)^2 \\
&+ \frac{1}{4}\big(te^t + te^{-t} - e^t + e^{-t}\big)(A-I)^2(A+I) \\
u(t) &= e^t \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + te^t \begin{bmatrix} -2 \\ 2 \\ -4 \\ 0 \end{bmatrix} + \big(\frac{t}{2}e^t - \frac{1}{4}e^t + \frac{1}{4}e^{-t}\big) \begin{bmatrix} 8 \\ -4 \\ 4 \\ -4 \end{bmatrix} \\
&+ \frac{1}{4}\big(te^t + te^{-t} - e^t + e^{-t}\big) \begin{bmatrix} 4 \\ 0 \\ -8 \\ -4 \end{bmatrix}
\end{aligned}
$$

5.7.6. (a) $\sin A = \frac{1}{3}\sin 4 \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} - \frac{1}{3}\sin 1 \begin{bmatrix} -2 & -1 \\ -2 & -1 \end{bmatrix}$,
$\cos A = \frac{1}{3}\cos 4 \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} - \frac{1}{3}\cos 1 \begin{bmatrix} -2 & -1 \\ -2 & -1 \end{bmatrix}$,
$\cos^2 A + \sin^2 A = I$. (b) Já.

6.2 Skilgreiningar og helstu reiknireglur

6.2.1. a) $\widehat{f}(\xi) = 2(\sin\xi)/\xi$, b) $\widehat{f}(\xi) = (ie^{-i\xi}/\xi + e^{-i\xi}/\xi^2 - 1/\xi^2)$,
c) $\widehat{f}(\xi) = 1/\xi^2 - i/\xi - e^{-i\xi}/\xi^2$, d) $\widehat{f}(\xi) = 2(1-\cos\xi)/\xi^2$,
e) $\widehat{f}(\xi) = 4(\sin\xi - \xi\cos\xi)/\xi^3$, f) $\widehat{f}(\xi) = c(e^{-ia\xi} - e^{-ib\xi})/i\xi$,
g) $\widehat{f}(\xi) = \dfrac{(b-a)e^{-ic\xi}}{(b-c)(c-a)\xi^2} - \dfrac{e^{-ia\xi}}{(c-a)\xi^2} - \dfrac{e^{-ib\xi}}{(b-c)\xi^2}$.

6.3 Andhverf Fourier–ummyndun

6.3.2. a) $\widehat{f}(\xi) = \dfrac{\pi}{\sqrt{2}}e^{-|\xi-1|/\sqrt{2}}$, b) $\widehat{f}(\xi) = \sqrt{\pi}(\frac{1}{16}\xi^4 - \frac{3}{4}\xi^2 + \frac{3}{4})e^{-\xi^2/4}$,
c) $\widehat{f}(\xi) = \sqrt{\dfrac{\pi}{\alpha}}e^{-(\xi-i\beta)^2/4\alpha}$, d) $\widehat{f}(\xi) = -4i\xi/(1+\xi^2)^2$,
e) $\widehat{f}(\xi) = -i\pi\mathrm{sign}(\xi)e^{-|\xi|}$, f) $\widehat{f}(\xi) = \pi(1-|\xi|)e^{-|\xi|}/2$, g) $\widehat{f}(\xi) = \pi(1+|\xi|)e^{-|\xi|}/2$, h) $\widehat{f}(\xi) = 2(\xi^2+2)/(\xi^4+4)$,
i) $\widehat{f}(\xi) = 2(2-\xi^2)/(\xi^4+4)$, j) $\widehat{f}(\xi) = \frac{1}{4}\pi e^{-|\xi|}(\cos\xi + \sin|\xi|)$.

6.3.3. a) $f(x) = 4/\pi(x^2+4x+20)$, b) $f(x) = (6+ix)e^{3ix-x^2/4}/4\sqrt{\pi}$,
c) $f(x) = (1-x^2)/\pi(1+x^2)^2$.

6.3.5. Nei, því fallið $1-\sin\xi$ stefnir ekki á núll í óendanlegu, eins og Fourier-myndir eiga að gera samkvæmt hjálparsetningu Riemanns og Lebesgues.

6.4 Földun og Fourier–ummyndun

6.4.1. a) $u(x) = 2e^{-2x^2}/\sqrt{\pi}$, b) $u(x) = \pm\sqrt{2}\pi^{-\frac{1}{4}}e^{-2x^2}$.
c) $u(x) = (e^{-|x|} - e^{-\sqrt{3}|x|})\mathrm{sign}(x)$.

6.4.2. $n\pi^{n-1}/(n^2+x^2)$.

6.4.3. $3\pi/(9+4x^2)$.

6.4.4. π.

6.5 Afleiðujöfnur og Fourier–ummyndun

6.5.2. $\widehat{f}(\xi) = \pi e^{-|\xi|}/(i\xi + 1 + e^{2i\xi})$.

6.5.3. a) $E(x) = H(x)xe^{-x}$,
$u(x) = xe^{-x}\int_{-\infty}^{x} e^y f(y)\,dy - e^{-x}\int_{-\infty}^{x} ye^y f(y)\,dy$,
$u(x) = \frac{1}{6}H(x)x^3e^{-x}$, ef $f(x) = H(x)xe^{-x}$.
b) $E(x) = -\frac{1}{2}e^{-|x|}$,
$u(x) = -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^y f(y)\,dy - \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-y} f(y)\,dy$,
$u(x) = -\frac{1}{8}e^{-|x|} - \frac{1}{4}H(x)e^{-x}(x+x^2)$.

6.6 Plancherel–jafnan

6.6.1. $\pi/2$

6.6.3. $\pi/2$.

6.9 Fourier–ummyndun af dreififöllum og grunnlausnir

6.9.1. a) $-i\pi(\delta_1 - \delta_{-1})$, b) $2\pi i^k\delta^{(k)}$, c) $2\pi\sum_{k=0}^{6}\binom{6}{k}i^k\delta^{(k)}$.

6.9.2. a) $(i\xi)^k e^{-ia\xi}$, b) $-2i\sin(a\xi)$, c) $2\cos(a\xi)$.

7.1 Skilgreiningar og nokkrar reiknireglur

7.1.2. a) $\mathcal{L}f(s) = 2/s(s^2+4)$, $\{s\in\mathbb{C}; s\neq 0, \pm 2i\}$,
b) $\mathcal{L}f(s) = \frac{1}{2}\sqrt{\pi}s^{-3/2} + 3s^{-2}$, $\mathbb{C}\setminus\{s\in\mathbb{R}; s\leq 0\}$,
c) $\mathcal{L}f(s) = s^{-1} + 3s^{-2} + 6s^{-3} + 6s^{-4}$, $\{s\in\mathbb{C}; s\neq 0\}$,
d) $\mathcal{L}f(s) = (s^2-4)/(s^2+4)^2$, $\{s\in\mathbb{C}; s\neq \pm 2i\}$,
e) $\mathcal{L}f(s) = 1/(s-1)^2$, $\{s\in\mathbb{C}; s\neq 1\}$,
f) $\mathcal{L}f(s) = \frac{3}{4}\sqrt{\pi}(s-2)^{-\frac{5}{2}}$, $\mathbb{C}\setminus\{s\in\mathbb{R}; s\leq 2\}$.

7.1.3. a) $\mathcal{L}f(s) = s^{-1}(e^{-as} - e^{-bs})$, b) $\mathcal{L}f(s) = 1/s(1-e^{-s})$,
c) $\mathcal{L}f(s) = 1/s(1+e^{-s})$, d) $\mathcal{L}f(s) = (1-e^{-s})/(s(1+e^{-s})) = s^{-1}\tanh(s/2)$.

7.1.5. a) $\mathcal{L}f(s) = s^{-2} - e^{-s}/(s(1-e^{-s}))$, b) $\mathcal{L}f(s) = s^{-2}\tanh(s/2)$.

7.1.6. $\mathcal{L}f(s) = k/((s^2+k^2)(1-e^{-\pi s/k}))$.

7.2 Andhverfa Laplace–ummyndunin

7.2.1. a) $\mathcal{L}^{[-1]}F(t) = \frac{1}{2}t^3$, b) $\mathcal{L}^{[-1]}F(t) = 3\cos 2t + \frac{1}{2}\sin 2t$,
c) $\mathcal{L}^{[-1]}F(t) = \frac{3}{5}\sinh 5t - 10\cosh 5t$,
d) $\mathcal{L}^{[-1]}F(t) = 2H(t-3)$, $H =$ Heaviside-fallið,
e) $\mathcal{L}^{[-1]}F(t) = H(t-2)\big(e^{-(t-2)} - e^{-2(t-2)} - (t-2)e^{-2(t-2)}\big)$,
f) $\mathcal{L}^{[-1]}F(t) = \frac{1}{3}H(t-1)\sin 3(t-1) + (1+\cos 3t)$.

7.3 Laplace–ummyndunin og upphafsgildisverkefni

7.3.1. a) $u(t) = \frac{1}{4}e^{-3t}\big(8\cos 4t + 9\sin 4t\big)$,
b) $u(t) = \frac{1}{8}\big(-2t + \sinh 2t\big)$,
c) $u(t) = \frac{1}{50}\big((5t-1)e^{-t} + e^{-2t}(\cos 3t + 32\sin 3t)\big)$,
d) $u(t) = \frac{1}{4}\big(3e^t + e^{-t}\cos 2t - e^{-t}\sin 2t\big)$,
e) $u(t) = 5e^t + 3te^{2t} - 4e^{2t}$.

7.3.2. $u_1(t) = e^t$, $u_2(t) = e^{-t}$, $u_3(t) = e^t - e^{-t}$.

7.3.3. $u(t) = \frac{2}{3}\sin t - \frac{1}{3}\sin 2t$, $v(t) = \frac{2}{3}\cos t + \frac{1}{3}\cos 2t$.

7.3.4. $u_1(t) = \frac{1}{5}\sin t + \frac{2}{5}\cos t + \frac{4}{5\sqrt{6}}\sin(\sqrt{6}t) - \frac{2}{5}\cos(\sqrt{6}t)$,
$u_2(t) = \frac{2}{5}\sin t - \frac{1}{5}\cos t + \frac{3}{5\sqrt{6}}\sin(\sqrt{6}t) + \frac{1}{5}\cos(\sqrt{6}t)$.

7.4 Green–fallið og földun

7.4.1. a) $g(t) = (\sin\omega t)/\omega$, b) $g(t) = \frac{1}{10}\sin 2t - \frac{1}{15}\sin 3t$,
c) $g(t) = \frac{1}{8}e^t - \frac{1}{8}e^{-t}\cos 2t - \frac{1}{8}e^{-t}\sin 2t$,
d) $g(t) = \frac{1}{5}te^{-t} + \frac{4}{25}e^{-t} - \frac{4}{25}e^t\cos t + \frac{3}{25}e^t\sin t$,
e) $g(t) = \frac{1}{10}\big(3\cos t + \sin t - 5e^t + 2e^{2t}\big)$,
f) $g(t) = \frac{1}{2}\big(te^t - e^t + \cos t\big)$.

7.4.2. a) $u(t) = \frac{1}{2}\big(\sin t - t\cos t\big)$, b) $u(t) = 1 + 2t^2$, c) $u(t) = \pm\sqrt{6}te^t$.

7.6 Laplace-ummyndun af dreififöllum

7.6.1. a) $u(t) = H(t-a)(1-\cos(t-a)) - H(t-b)(1-\cos(t-b))$,
b) $u(t) = \sum_{n=0}^{\infty} H(t-n)(1-(t-n-1)e^{n-t})$,
c) $u(t) = \sum_{n=0}^{\infty} (-1)^n H(t-n)\frac{1}{4}(e^t - e^{-t} - 2te^{-t})$,
d) $u(t) = \sum_{n=0}^{\infty} H(t-n)(1-\cos(t-n))$,
e) $u(t) = \max\{\sin t, 0\}$,
f) $u(t) = (n+1)\sin t$, $2n\pi < t < 2(n+1)\pi$.

8.2 Fourier–raðir af lotubundnum föllum

8.2.1. Allir stuðlar eru núll nema: a) $a_0 = 1$, $a_2 = \frac{1}{2}$, $c_0 = \frac{1}{2}$, $c_2 = c_{-2} = \frac{1}{4}$,
b) $b_1 = 1$, $c_1 = -\frac{i}{2}$, $c_{-1} = \frac{i}{2}$.
c) $a_0 = \frac{3}{4}$, $a_2 = \frac{1}{2}$, $a_4 = \frac{1}{8}$, $c_0 = \frac{3}{8}$, $c_2 = c_{-2} = \frac{1}{4}$, $c_4 = c_{-4} = \frac{1}{16}$.
d) $a_0 = \frac{1}{4}$, $a_4 = -\frac{1}{8}$, $c_0 = \frac{1}{8}$, $c_4 = c_{-4} = -\frac{1}{16}$.
e) $a_{2n} = \frac{1}{2^{19}}\binom{20}{10-n}$, $n = 0, \dots, 10$, $c_{2n} = \frac{1}{2^{20}}\binom{20}{10-n}$, $n = -10, \dots, 10$.

8.2.2. $f(x) = e^{ix-1}/(1 - e^{ix-1})^2$.

8.2.3. $f(x) = c_1 e^{ix} + c_{-1}e^{-ix}$, $c_1, c_{-1} \in \mathbb{C}$.

8.4 Andhverfuformúla Fouriers

8.4.2. $u(x) = \dfrac{2\sinh\pi}{\pi}\left(\dfrac{1}{2} + \sum_{k=1}^{\infty}\dfrac{(-1)^k}{1+k^2}\cos kx\right)$, $\sum_{k=1}^{\infty}\dfrac{1}{1+k^2} = \frac{1}{2}\pi\coth\pi - \frac{1}{2}$.

8.4.3. $u(x) = \dfrac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}\cos nx$.

8.4.4. $f(x) = \dfrac{\alpha\sin\pi\alpha}{\pi}\sum_{n=-\infty}^{+\infty}\dfrac{(-1)^n}{\alpha^2 - n^2}e^{inx}$.

8.4.5. $f(x) = \dfrac{1}{\pi} + \dfrac{1}{2}\cos x - \dfrac{2}{\pi}\sum_{k=1}^{\infty}\dfrac{(-1)^k}{4k^2-1}\cos\big(2kx\big)$,
$\sum_{k=1}^{\infty}\dfrac{(-1)^k}{4k^2-1} = \dfrac{1}{2} - \dfrac{\pi}{4}$, $\sum_{k=1}^{\infty}\dfrac{1}{4k^2-1} = \dfrac{1}{2}$.

8.5 Parseval–jafnan

8.5.1. $\sum_{n=1}^{\infty} 1/n^4 = \pi^4/90$.

8.5.2. $c_0 = 0$, $c_n = -6i(-1)^n/n^3$, $n \neq 0$, $\sum_{n=1}^{\infty} 1/n^6 = \pi^6/945$.

8.5.3. $c_n = (e^{2\pi i\alpha} - 1)/2\pi i(\alpha - n)$.

8.7 Fourier–raðir og afleiðujöfnur

8.7.5. $\varphi(x) = \sum_{n=0}^{\infty}\dfrac{(-1)^n 8}{(2n+1)^2\pi^2}\sin\big((2n+1)\pi x\big)$.
$u(x,t) = e^{-t}\sum_{n=0}^{\infty}\dfrac{(-1)^n 8}{(2n+1)^2\pi^2}\left(\cos\big(\omega_{2n+1}t\big) + \dfrac{\sin\big(\omega_{2n+1}t\big)}{\omega_{2n+1}}\right)\sin\big((2n+1)\pi x\big)$,
$\omega_n = \sqrt{n^2\pi^2 - 1}$.

8.7.6. $u(x,t) = \sum_{n=0}^{\infty} a_n e^{-\kappa n^2\pi^2/L^2}\cos\big(n\pi x/L\big)$
a) $a_n = \dfrac{2L^2}{n^2\pi^2}\big((1-(-1)^n)T_0 + ((-1)^n - 1)T_1\big)$.
b) $a_n = \dfrac{4L^2}{n^2\pi^2}\big(1 - 2\cos(n\pi/2) + (-1)^n\big)\big(T_0 - T_1\big)$.

9.1 Virkjar af Sturm–Liouville–gerð

9.1.1. Bessel: $-\frac{1}{x}\left(\frac{d}{dx}\left(x\frac{du}{dx}\right)+x\left(1-\frac{n^2}{x^2}\right)u\right)$.
Chebychev: $-\sqrt{1-x^2}\left(\frac{d}{dx}\left(\sqrt{1-x^2}\frac{du}{dx}\right)+\frac{n^2}{\sqrt{1-x^2}}u\right)$.
Hermite: $-e^{x^2}\left(\frac{d}{dx}\left(e^{-x^2}\frac{du}{dx}\right)+2ne^{-x^2}u\right)$.
Laguerre: $-e^{x}\left(\frac{d}{dx}\left(xe^{-x}\frac{du}{dx}\right)+ne^{-x}u\right)$.
Legendre: $-\left(\frac{d}{dx}\left((1-x^2)\frac{du}{dx}\right)+n(n+1)u\right)$.

9.1.2. $\alpha\delta-\beta\gamma=1$.

9.2 Eigingildisverkefni af Sturm–Liouville–gerð

9.2.1. Almenn lausn: $u(x)=Ae^{i\beta x^2}+Be^{-i\beta x^2}$, $\lambda=\beta^2/4$, $\mathrm{Re}\beta>0$.
Eigingildi: $\lambda_n=4n^2\pi^2/9$, $n=1,2,3\dots$.
Tilsvarandi eiginföll: $u_n(x)=\sin\big(n\pi(x^2-1)/3\big)$.
Innfeldi: $\langle u,v\rangle=\int_1^2 u(x)\overline{v(x)}x\,dx$.

9.2.7.

$$u(x,t)=\sum_{n=1}^{\infty}e^{-\kappa\beta_n^2 t/L^2}\cos\big(\beta_n x/L\big),$$

þar sem β_n eru jákvæðar lausnir jöfnunnar $\tan\beta=hL/\beta$ og

$$c_n=\frac{2h}{hL+\sin^2\beta_n}\int_0^L\varphi(x)\cos\big(\beta_n x/L\big)\,dx.$$

9.2.8.

$$u(x,y)=\sum_{n=1}^{\infty}c_n\sinh\big((\beta_n/L)(L-x)\big)\cos\big(\beta_n y/L\big),$$

þar sem β_n eru jákvæðar lausnir jöfnunnar $\tan\beta=hL/\beta$ og

$$c_n=\frac{2h}{\sinh\beta_n\big(hL+\sin^2\beta_n\big)}\int_0^L g(y)\cos\big(\beta_n y/L\big)\,dy.$$

10.1 Vegheildun og Green–setningin

10.1.2. a) $ir(2+\pi r)$, r er geisli hringsins, b) $(2+11i)/3$, c) 0,
d) $(1+i)\sqrt{2}\big(\sqrt{\sqrt{2}+1}-\sqrt{\sqrt{2}-1}\big)$, e) $4i(\ln 2-1)$, f) $i\pi$, g) $(1+2i)e^{2+i}+8e^{1+3i}$.

10.2 Cauchy–setningin og Cauchy–formúlan

10.2.1. a) $2\pi/(1-a^2)$, b) $5\pi/12$.

10.2.2. a) $a_n(f) = 2a^n/(1-a^2)$, $b_n(f) = 0$.

10.2.3. a) $2\pi/\sqrt{3}$, b) $\pi/\sqrt{2}$, c) $2\pi/3$.

10.2.4. a) $\widehat{f}(\xi) = 2\pi\exp(-\frac{1}{2}(\sqrt{3}|\xi| - i\xi))$,
c) $\widehat{f}(\xi) = \frac{1}{3}\pi\big(e^{-|\xi|} + e^{-|\xi|/2}(\cos(\frac{\sqrt{3}}{2}\xi) + \sin(\frac{\sqrt{3}}{2}|\xi|))\big)$.

10.6 Einangraðir sérstöðupunktar

10.6.1. $S(z) = z - \dfrac{1}{3\cdot 3!}z^3 + \dfrac{1}{5\cdot 5!}z^5 - \dfrac{1}{7\cdot 7!}z^7 + \dfrac{1}{9\cdot 9!}z^9 + \cdots$,
$E(z) = z + \dfrac{1}{2\cdot 2!}z^2 + \dfrac{1}{3\cdot 3!}z^3 + \dfrac{1}{4\cdot 4!}z^4 + \dfrac{1}{5\cdot 5!}z^5 + \cdots$,
$L(z) = z - \dfrac{1}{2^2}z^2 + \dfrac{1}{3^2}z^3 - \dfrac{1}{4^2}z^4 + \dfrac{1}{5^2}z^5 + \cdots$.

10.6.2. a) $z = 0$, afmáanlegur,
b) $z = 0$ er afmáanlegur, $z = 1$ er skaut af stigi 1, $\text{Res}(f, 1) = e - 1$.
c) $z = 0, \pi$ afmáanlegir, $z = n\pi$, $n \neq 0, 1$, skaut af stigi 1, $\text{Res}(f, n\pi) = (-1)^n n(n-1)\pi^2$
d) $z = 0$, verulegur, $\text{Res}(f, 0) = 1$

10.7 Leifasetningin

10.7.1. $\binom{2n}{n}\frac{2\pi}{2^{2n}}$.

11.1 Útreikningur á leifum

11.1.1. a) $\alpha = 1$ er skaut af stigi 2 $\text{Res}(f, 1) = 5\pi/4$, $\alpha = 2$ er skaut af stigi 1, $\text{Res}(f, 2) = -\pi$ og $\alpha = 3$ er afmáanlegur sérstöðupunktur.
b) $\alpha = 0$ er skaut af stigi 4 og $\text{Res}(f, 0) = 1/6$.
c) $\alpha = 0$ er verulegur sérstöðupunktur og $\text{Res}(f, 0) = 0$.
d) $\alpha = 0, 1, 2$ eru skaut af stigi 1, $\text{Res}(f, 0) = \frac{1}{2}$, $\text{Res}(f, 1) = -1$, $\text{Res}(f, 2) = \frac{1}{2}$.
e) $\alpha = \frac{1}{2}\sqrt{2}(\pm 1 \pm i)$, er skaut af stigi 1, $\text{Res}(f, \alpha) = -\alpha/4$.
f) $\alpha = \frac{1}{2}(\pm\sqrt{3} \pm i), \pm i$ eru skaut af stigi 1, $\text{Res}(f, \alpha) = -\alpha/6$.
g) $\alpha = (n + \frac{1}{2})\pi$ er skaut af stigi 1, $\text{Res}(f, (n + \frac{1}{2})\pi) = -1$.
h) $\alpha = 0$ er verulegur sérstöðupunktur, $\text{Res}(f, 0) = \sum\limits_{k=0}^{\infty} \dfrac{1}{k!(k+1)!} \approx 1.59$.
i) $\alpha = 0$ er skaut af stigi 2, $\text{Res}(f, 0) = 0$.
$\alpha = n\pi$, $n \in \mathbb{Z} \setminus \{0\}$, er skaut af stigi 1, $\text{Res}(f, n\pi) = \dfrac{(-1)^{n+1}}{\sinh(n\pi)}$,
$\alpha = in\pi$, $n \in \mathbb{Z} \setminus \{0\}$, er skaut af stigi 1, $\text{Res}(f, in\pi) = \dfrac{(-1)^{n+1}i}{\sinh(n\pi)}$,
j) $\alpha = i(n + \frac{1}{2})$, $n \in \mathbb{Z}$, er skaut af stigi 1, $\text{Res}(f, i(n + \frac{1}{2})) = i(n + \frac{1}{2})/\pi$.

11.1.2. a) $-i\pi$, b) $i\pi(\cosh 1)/3$.

11.2 Heildi yfir einingarhringinn

11.2.1. $2\pi/2a(1+a^2)^{\frac{1}{2}}$.

11.2.2. $\pi/\sqrt{2}$.

11.3 Heildi yfir raunásinn

11.3.1. a) $3\pi/8$, b) $\pi/2$, c) $5\pi/144$, d) $\pi(\sqrt{2+\sqrt{2}}-\sqrt{2-\sqrt{2}})/4$,
e) π/e, f) π, g) π, h) $\pi e^{-\sqrt{2}/2}\sin(\sqrt{2}/2)$.

11.4 Útreikningur á Fourier–myndum

11.4.1. a) $\widehat{f}(\xi)=\frac{1}{8}\pi e^{-|\xi|}\big(\xi^2+3|\xi|+3\big)$,
b) $\widehat{f}(\xi)=\dfrac{\pi}{3}\bigg(e^{-|\xi|}+e^{-|\xi|/2}\big(\cos(\sqrt{3}\xi/2)+\sqrt{3}\sin(\sqrt{3}|\xi|/2)\big)\bigg)$.
c) $\widehat{f}(\xi)=-\frac{1}{2}\text{sign}(\xi)\pi e^{(-1+i)|\xi|}(1+(1+i)|\xi|)$.
d) $\widehat{f}(\xi)=\pi\sqrt{2}e^{-|\xi|/\sqrt{2}}\cos(\xi/\sqrt{2})$.
e) $\widehat{f}(\xi)=\frac{1}{3}\pi i\text{sign}(\xi)\bigg(e^{-|\xi|}-e^{-|\xi|/2}\cos(\frac{\sqrt{3}}{2}\xi)-e^{-|\xi|/2}\sqrt{3}\sin(\frac{\sqrt{3}}{2}|\xi|)\bigg)$.
f) Setjum $\alpha_j=e^{ik2\pi/5}$, $k=1,2,3,4$. Þá er

$$\widehat{f}(\xi)=\begin{cases}\dfrac{2\pi i}{5}\big(e^{-i\alpha_1\xi}\alpha_1(\alpha_1-1)+e^{-i\alpha_2\xi}\alpha_2(\alpha_2-1)\big), & \xi\le 0,\\ -\dfrac{2\pi i}{5}\big(e^{-i\alpha_3\xi}\alpha_3(\alpha_3-1)+e^{-i\alpha_4\xi}\alpha_4(\alpha_4-1)\big), & \xi>0.\end{cases}$$

11.4.2. a) $\mathcal{F}^{[-1]}F(x)=\frac{1}{2}(-1+\text{sign}(x)i)e^{-|x|-ix}$.
b) $\frac{1}{20}(-2\text{sign}(x)+11i)(5+2\text{sign}(x)+6i)e^{-|x|-2xi}$.

11.5 Útreikningur á andhverfum Laplace–myndum

11.5.1. $G(t,\tau)=g(t-\tau)$, $g(t)=\frac{1}{2}\big(te^t-e^t+\cos t\big)$.

11.5.2. a) $\mathcal{L}^{-1}\{F\}(t)=(1-2t+\frac{1}{2}t^2)e^{-3t}$
b) $\mathcal{L}^{-1}\{F\}(t)=-1+\frac{1}{2}t^2+\cos t$,
c) $\mathcal{L}^{-1}\{F\}(t)=-\frac{i}{2}e^{(1-i)t}+\frac{(1-i)}{2}te^{2t}-\frac{i}{2}e^{2t}$.

11.5.3. $G(t,\tau)=g(t-\tau)$,
c) $g(t)=\frac{1}{2}\big(te^t-e^t+\cos t\big)$.
d) $g(t)=\frac{1}{5}te^{-t}+\frac{4}{25}e^{-t}-\frac{4}{25}e^t\cos t+\frac{3}{25}e^t\sin t$,