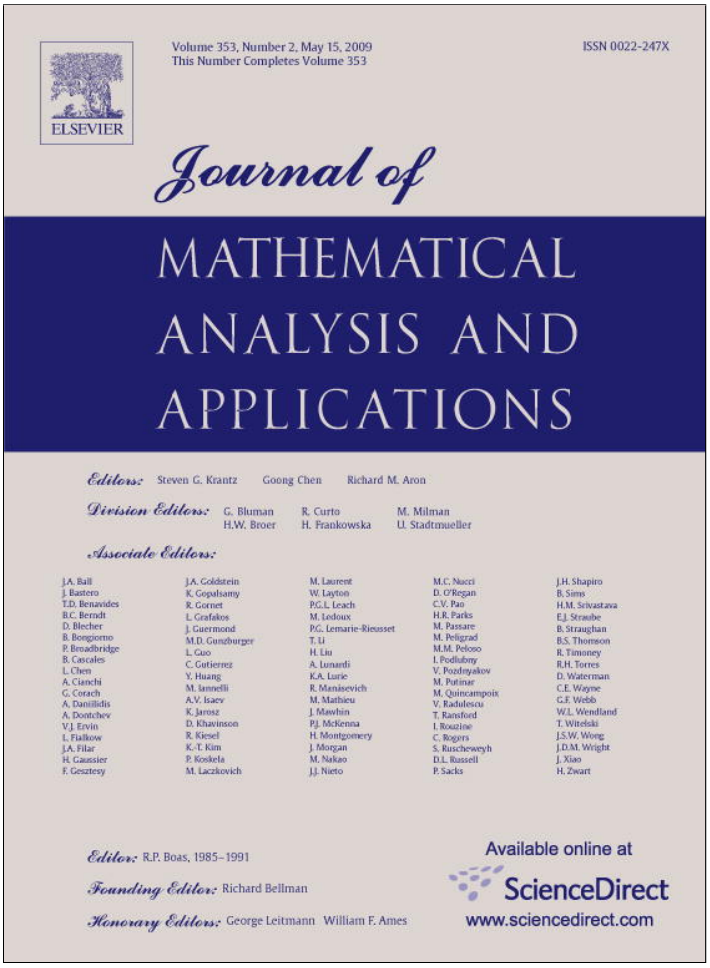
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A Stone–Weierstrass theorem for Banach function spaces satisfying a certain separation property

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ABSTRACT

We consider a strong lattice property for a Banach function space B on a compact Hausdorff space, which gives a general Stone–Weierstrass theorem for B. We also study the relation of this theorem and its proof to a certain decomposition of an associated compactification, and to another lattice-like property.

1. Introduction

A long tradition of inquiry seeks sufficient sets of conditions on a linear subspace B of $C(X)$, the space of continuous real-valued functions on a compact Hausdorff space X, in order that B be (uniformly) dense in, or even equal to, $C(X)$. The most prominent results along these lines are the Stone–Weierstrass theorems, in which the key hypothesis (beyond point separation and containing the constant functions) is either that B be a lattice or that B be an algebra, in both cases under pointwise operations, and the conclusion is density. The lattice and algebra conditions can be reformulated to assert that B is closed under composition with an appropriate continuous function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = |t|$ in the first case and $\varphi(t) = t^2$ in the second. In 1963 K. de Leeuw and Y. Katznelson [9] showed that the density conclusion can be achieved if φ is *any* non-affine continuous function on an interval.

About the same time, J. Wermer [12] showed that if $B = \Re(A)$ consists of the real parts of the functions in a (complex) uniform algebra A and B is itself an algebra, then $B = C(X)$ and $A = C_{\mathbb{C}}(X)$ (the space of continuous complex-valued functions on X). Since $B = \Re(A)$ is a Banach space in a natural quotient norm, the following broad problem (precise definitions below) presents itself: What extra condition(s) on a Banach function space B and/or a continuous function φ that operates on it force the conclusion $B = C(X)$? Our main theorem gives a separation condition on B that guarantees that $B = C(X)$ if there is *any* non-affine continuous function that operates on B. In Section 2 we present the sorts of separation conditions on a Banach function space that will interest us, and in Section 3 we prove the main theorem (Theorem 1). Section 4 is devoted to finer structures than those we used in our proofs; these can be used in an alternative development of our main result.

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2. Separation conditions

A *Banach function space* $(B, \|\cdot\|)$ on a compact Hausdorff space X is a subspace of $C(X)$ which contains the constant functions and separates the points of X, and whose norm $\|\cdot\|$ dominates the supremum norm $\|\cdot\|_\infty$. (Our scalars will be real, unless, as in the previous paragraph, complex scalars are explicitly indicated.)

We are interested in a special separation condition for B, introduced by A.J. Ellis [4], involving pairs of disjoint compact subsets of X:

> There are a positive number M and a natural number N such that given any pair F, G of disjoint compact subsets of X there are $b_i, c_i \in B$, with $\|b_i\|, \|c_i\| \leqslant M$, $1 \leqslant i \leqslant N$, such that
>
> $$b_1 \wedge \cdots \wedge b_N - c_1 \wedge \cdots \wedge c_N > 1 \quad \text{on } F \quad \text{and} \quad b_1 \wedge \cdots \wedge b_N - c_1 \wedge \cdots \wedge c_N < 0 \quad \text{on } G.$$

When $N = 1$ this can only happen if $B = C(X)$ and the norms $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent. We give a proof for the convenience of the reader. Suppose that the above holds with $N = 1$. Let $u \in C(X)$ with $\|u\|_\infty \leqslant 1$ and put $F = \{x \in X\colon\ u(x) \geqslant 2/3\}$ and $G = \{x \in X\colon\ u(x) \leqslant -2/3\}$. Then there is $b \in B$ with $\|b\| \leqslant 4M$ such that $b > 1$ on F and $b < -1$ on G. Letting $b_1 = \frac{b}{24M}$, by a simple calculation we have

$$\|u - b_1\|_\infty \leqslant 1 - \delta,$$

where $\delta = \frac{1}{24M}$. Applying this argument successively, we can find a sequence $\{b_n\}$ of functions in B with $\|b_n\| \leqslant 6^{-1}(1-\delta)^{n-1}$,

$$\left\| u - \sum_{k=1}^{n} b_k \right\|_\infty \leqslant (1-\delta)^n.$$

It follows that $\sum_{n=1}^\infty b_n \in B$ and $u = \sum_{n=1}^\infty b_n$ since the original norm $\|\cdot\|$ dominates the supremum norm. We remark that point separation means that $lat(B)$, the lattice generated by B, is dense in $C(X)$.

A word is in order about $lat(B)$. If B is a non-empty *subset* of $C(X)$, $lat(B)$ is of course a lattice of functions. Furthermore, if B is a vector space of functions, then it is not hard to see that $lat(B)$ is also a vector space, so is a *vector lattice*, and that $lat(B)$ consists precisely of the functions of the form

$$b_1 \wedge \cdots \wedge b_n - c_1 \wedge \cdots \wedge c_n, \tag{$\star$}$$

where n is a natural number and b_i and c_i are elements of B. In this difference, $\wedge$ can be replaced by $\vee$ in both terms, or in just one term if the subtraction is replaced by addition.

There is a quite different characterization of the separation property. Let $\tilde{X} = \beta(\mathbb{N} \times X)$ denote the Stone–Čech compactification of $\mathbb{N} \times X$, where the space $\mathbb{N}$ of natural numbers is given the discrete topology. The space $\ell^\infty(C(X))$ of all $\|\cdot\|_\infty$-bounded sequences of functions in $C(X)$ has a natural representation as $C(\tilde{X})$, and *via* this representation the space $\ell^\infty(B)$ of all $\|\cdot\|$-bounded sequences of functions in B, a subspace of $\ell^\infty(C(X))$, can be considered to be a subspace $\tilde{B}$ of $C(\tilde{X})$. A typical element of $\tilde{B}$ is $\tilde{b} = (b_n)$ where $\|\tilde{b}\| = \sup_n \|b_n\| < \infty$. (Notice that our notation gives $\widetilde{C(X)} = C(\tilde{X})$.) The above separation condition is equivalent to the condition that $\tilde{B}$ separates the points of $\tilde{X}$ in the same way as [4] for the case of uniformly closed spaces, and then B is said to be *ultraseparating* on X, a notion first introduced and investigated by A. Bernard in his seminal paper [1]. Naturally, this is equivalent to the density of $lat(\tilde{B})$ in $C(\tilde{X})$.

We are mainly interested in a slightly stronger separation condition. For $x \in X$ let B_x denote the space of functions in B vanishing at x. The strengthening is the following *local* separation condition at each $x \in X$:

> There are a positive number M and a natural number N such that given any pair F, G of disjoint compact subsets of $X \backslash \{x\}$ there are $b_i, c_i \in B_x$, with $\|b_i\|, \|c_i\| \leqslant M$, $1 \leqslant i \leqslant N$, such that
>
> $$b_1 \wedge \cdots \wedge b_N - c_1 \wedge \cdots \wedge c_N > 1 \quad \text{on } F \quad \text{and} \quad b_1 \wedge \cdots \wedge b_N - c_1 \wedge \cdots \wedge c_N < 0 \quad \text{on } G.$$

Before giving examples we give two equivalent characterizations of the local separation property above. The corresponding global versions are obtained by removing the references to x. The first additional characterization is this:

> There are a positive number M and a natural number N such that given any pair F, G of disjoint compact subsets of $X \backslash \{x\}$ there are partitions $F = \bigcup_{i=1}^N F_i$ and $G = \bigcup_{j=1}^N G_j$ and functions $b_{ij} \in B_x$ with $\|b_{ij}\| \leqslant M$, $1 \leqslant i, j \leqslant N$, such that
>
> $$b_{ij} > 1 \quad \text{on } F_i \quad \text{and} \quad b_{ij} < -1 \quad \text{on } G_j$$
>
> for any pair i, j.

The second characterization is the local version of ultraseparation. We use a bar over a set to indicate closure in $\tilde{X}$, and if $Y \subset \tilde{X}$ we write $C(\tilde{X})_Y$ for the space of functions in $C(\tilde{X})$ that are identically zero on Y. The characterization is:

$$lat(\widetilde{B_x}) \text{ is dense in } \widetilde{C(X)_x} = C(\tilde{X})_{\overline{\mathbb{N} \times \{x\}}}.$$

For the equivalence of the various conditions we refer to [4]. There only the $\|\cdot\|$-norm and the full subspaces (without the subscripts) are considered, but the proofs are the same in the local situations. Because the sequences $\tilde{c} = (c_n)$ where each c_n is a constant function separate the points of $\overline{\mathbb{N}\times\{x\}}$, it follows that if B satisfies the local separation condition at every point x, then B satisfies the global separation condition, that is, B is ultraseparating.

The above condition on $lat(\widetilde{B_x})$ is equivalent to what we may call the $(-+)$ *condition* for $\widetilde{B_x}$ on $\tilde{X}$: If p and q are two distinct points of $\tilde{X}\backslash\overline{\mathbb{N}\times\{x\}}$, there is $\tilde{b}\in\widetilde{B_x}$ such that $\tilde{b}(p) > 0$ and $\tilde{b}(q) \leqslant 0$. For on the one hand, if the $(-+)$ condition fails for some p and q, then either $\tilde{b}(p) = 0$ for all $\tilde{b}\in\widetilde{B_x}$ or there is a non-negative constant k such that $\tilde{b}(q) = k\tilde{b}(p)$ for all $\tilde{b}\in\widetilde{B_x}$, in either case implying that every $\tilde{u}\in lat(\widetilde{B_x})$ satisfies the same condition, so density fails for $lat(\widetilde{B_x})$. On the other hand, if the $(-+)$ condition is satisfied for some p and q then values of functions in $lat(\widetilde{B_x})$ at p and q interpolate every pair of real numbers, and if this is the case for all p and q then the usual proof of the lattice version of the Stone–Weierstrass theorem gives the desired density.

The following example is due to Hatori [7].

Example 1. Let X be the subset of the real line given by $X = \{\pm 1/n\colon\ n\in\mathbb{N}\}\cup\{0\}$ and let B consist of the linear span of the constant functions and those continuous functions b on X that satisfy the condition $b(1/n) = (1/2)b(-1/n)$ for all $n\in\mathbb{N}$; alternatively, B consists of those continuous functions c such that $2c(x) = c(-x) + c(0)$ for $0\leqslant x\in X$. Endow B with the supremum norm.

It is easy to see that B satisfies the second global separation condition (for example with $N = 3$ and any $M > 3$) but not the local version at $x = 0$.

Example 2. If A is a (complex) uniform algebra on X, then $B = \Re(A)$, the space of real parts of functions in A, is a Banach function space in the quotient norm

$$\|b\| = \inf\bigl\{\|b + ic\|_\infty\colon\ c\in B,\ b + ic\in A\bigr\}.$$

A is said to *approximate in modulus* on X if given $0\leqslant g\in C(X)$ and $\varepsilon > 0$, there is $f\in A$ such that $||f| - g| < \varepsilon$ on X. It is clear that then $\tilde{A}$ approximates in modulus on $\tilde{X}$, so separates the points of $\tilde{X}$. Moreover, given $x\in X$ and finitely many points $p_1,\ldots,p_n\in\tilde{X}\backslash\overline{\mathbb{N}\times\{x\}}$, there is $\tilde{g} = (g_n)\in\tilde{A}$ such that $|\tilde{g}(p_j)| > 1$ for all j, $|\tilde{g}(p_j) - \tilde{g}(p_k)| > 3$ for all j,k for which $j\neq k$, and $|\tilde{g}| < 1$ on $\overline{\mathbb{N}\times\{x\}}$. Letting $f_n = g_n - g_n(x)$ gives $\tilde{f}\in\widetilde{A_x}$ that takes different nonzero values at the different p_j. Because $\widetilde{A_x}$ is an algebra, it can interpolate *any* sequence of n complex values on the p_j, and so its real part $\widetilde{B_x}$ can interpolate any sequence of n real values on these points, a very strong form of the local separation condition.

Two important classes of uniform algebras approximate in modulus. Suppose A is *Dirichlet* on X, that is, $B = \Re(A)$ is dense in $C(X)$. Then the set of exponentials of functions in A is a subset of A whose moduli approximate all non-negative continuous functions uniformly. That B satisfies the *global* separation condition when A is Dirichlet was proven by Bernard in [1] by noting that such an A approximates in modulus. The second class consists of those A that contain sufficiently many unimodular functions to separate the points of X. For an A of this sort, functions of the form $(c_1 f_1 + \cdots + c_n f_n)/g$ where $n\in\mathbb{N}$, the c_j are complex constants, and $f_1,\ldots,f_n$, g are unimodular functions in A are a self-adjoint point-separating algebra of continuous functions on X, so are dense in $C_{\mathbb{C}}(X)$, so their moduli approximate all non-negative continuous functions uniformly; but the modulus of such a function is the modulus of its numerator, which is an element of A.

3. Operating functions and the main theorem

We now introduce the concept of an *operating function* for a Banach function space B on X. A function φ defined on an interval I of the real line is said to *operate* on B if $\varphi\circ b\in B$ whenever $b\in B$ and the composition is defined, *i.e.*, $b(X)\subset I$. Functions of the kind $\varphi(t) = \alpha t + \beta$, the *affine* functions, operate on any B, and these may be the only ones.

If Y is a non-empty compact subset of X, recall that $B|Y$, the space of restrictions $b|Y$ of functions $b\in B$ to Y, is itself a Banach function space (on Y) in the norm $\|u\| = \inf\{\|b\|\colon\ b\in B,\ b|Y = u\}$. If φ operates on B, it need not *a priori* be the case that it operates on $B|Y$, since it is possible that some $u\in B|Y$ whose range is in I may not have an extension $b\in B$ whose range is also in I; this turns out to be a minor technical detail. We can now state the main result of this note.

Theorem 1. *Let B be a Banach function space on a compact Hausdorff space X and suppose every $x\in X$ has a compact neighbourhood Y such that $B|Y$ satisfies one (and hence all) of the local separation conditions at x. If B has a continuous non-affine operating function φ then $B = C(X)$.*

The hypotheses on B force it to be ultraseparating on X, in which event the theorem has been proved in [10] in the case where φ is not affine on any subinterval of its domain. We will therefore assume that φ is affine on some non-degenerate subinterval.

For the proof we need special subsets of $\tilde{X}$. For $f\in C(X)$ let $\beta(x,f)$ be the set

$$\beta(x,f) = \bigl\{\xi\in\tilde{X}\colon\ (f)(\xi) = f(x)\bigr\}.$$

Here (f) is the element of $\ell^\infty(C(X))$ all of whose terms are f. Clearly $\mathbb{N}\times\{x\}$ is a subset of $\beta(x,f)$, hence so is its closure $\overline{\mathbb{N}\times\{x\}}$. If Y is a non-empty compact subset of X, $\tilde{Y}=\beta(\mathbb{N}\times Y)$ is naturally a compact subset of $\tilde{X}$, and if $x\in Y$ then $\beta(x,f|Y)=\beta(x,f)\cap\tilde{Y}$ for any continuous function f on X. The importance of these sets is due to the following local version of the so-called *Bernard's lemma* [1]:

Lemma 1. *Let B be a Banach function space on X. Suppose that, for a given $x\in X$ and $f\in C(X)$, it is the case that whenever $\mathcal{F}$ and $\mathcal{G}$ are disjoint compact subsets of $\beta(x,f)$ that do not meet $\overline{\mathbb{N}\times\{x\}}$ there is an element (b_n) of $\ell^\infty(B_x)$ such that $(b_n)>1$ on $\mathcal{F}$ and $(b_n)<0$ on $\mathcal{G}$. Then there is a compact neighbourhood K of x such that $B|K=C(K)$.*

Proof. The proof is modeled on one in [6] and [2]. Let $K_n=\{y\in X\colon |f(y)-f(x)|\leqslant 1/n\}$, a compact neighbourhood of x, and $\mathcal{K}_n=\{\xi\in\tilde{X}\colon |(f)(\xi)-f(x)|\leqslant 1/n\}$, so that $\beta(x,f)=\bigcap\mathcal{K}_n$. We will show that there are a natural number n_0 and a positive number M such that there is for every pair F, G of disjoint compact subsets of $K_{n_0}\backslash\{x\}$ a function b in B_x with $\|b\|\leqslant M$, $b>1$ on F and $b<0$ on G. Standard successive approximation arguments as before then show that $B_x|K_{n_0}=C(K_{n_0})_x$, hence $B|K_{n_0}=C(K_{n_0})$.

If no such n_0 and M exist, then for each natural number n there are disjoint compact subsets F_n, G_n of $K_n\backslash\{x\}$ such that $\|b\|>n$ if $b\in B_x$ satisfies $b>1$ on F_n and $b<0$ on G_n. Let

$$\mathcal{F}_n=\overline{\bigcup_{k=n}^{\infty}\big(\{k\}\times F_k\big)}\quad\text{and}\quad \mathcal{G}_n=\overline{\bigcup_{k=n}^{\infty}\big(\{k\}\times G_k\big)},$$

disjoint compact subsets of $\mathcal{K}_n\backslash\overline{\mathbb{N}\times\{x\}}$ for each $n\in\mathbb{N}$. Then $\mathcal{F}=\bigcap\mathcal{F}_n$ and $\mathcal{G}=\bigcap\mathcal{G}_n$ are disjoint compact subsets of $\beta(x,f)$ that do not meet $\overline{\mathbb{N}\times\{x\}}$, so by assumption there is (b_n) in $\ell^\infty(B_x)$ such that $(b_n)>1$ on $\mathcal{F}$ and $(b_n)<0$ on $\mathcal{G}$, and hence $(b_n)>1$ on some neighbourhood $\mathcal{U}$ of $\mathcal{F}$ and $(b_n)<0$ on some neighbourhood $\mathcal{V}$ of $\mathcal{G}$. We pick $n_1\in\mathbb{N}$ so that $\mathcal{F}_n\subset\mathcal{U}$ and $\mathcal{G}_n\subset\mathcal{V}$ for all $n\geqslant n_1$. If $n_2\in\mathbb{N}$ satisfies $n_2\geqslant n_1$ and $n_2\geqslant\sup_n\|b_n\|$, then $b_{n_2}>1$ on F_{n_2}, $b_{n_2}<0$ on G_{n_2}, and $\|b_{n_2}\|\leqslant n_2$, contrary to our choice of F_{n_2} and G_{n_2}. □

We now use the assumption that φ is affine on some non-degenerate subinterval of I. Composing φ with affine functions, we may assume that $I=(-1,1)$ and that φ maps I into I. Continuing composing with affine functions, we can construct operating functions φ_1 mapping I into I with $\varphi_1\geqslant 0$ and $\varphi_1(t)=0$ if and only if $t=0$, and φ_2 mapping I into I with $\varphi_2\geqslant 0$, $\varphi_2\equiv 0$ on $[0,1)$, and φ_2 is not identically zero on $(-\delta,0]$ for any $0<\delta<1$.

Lemma 2. *Let B be a Banach function space on X and suppose $x\in X$ has a compact neighbourhood Y such that $B|Y$ satisfies one (and hence all) of the local separation conditions at x. If B has a continuous non-affine operating function φ that is affine on some non-degenerate subinterval of its domain, then x has a compact neighbourhood K contained in Y such that $B|K=C(K)$.*

Proof. We assume as above that the domain of φ is $I=(-1,1)$, that φ maps I into itself, and that φ_1 and φ_2 are as described. We use the Baire category theorem as in [7] and obtain $0<\varepsilon<1, M>1, b_0\in B_x$, and a dense subset of the closed ε-ball of B_x such that, if ψ is any of the functions φ_1,φ_2 and $\varphi_1\circ\varphi_2$, $\psi(b_0+b)$ is in the closed M-ball of B_x whenever b is in the dense subset. Restricting to $\beta(x,b_0)$, we see that if for $t>0$ we denote by $\mathcal{B}^t$ the closed t-ball of $\ell^\infty(B_x)$ and by $\overline{\mathcal{B}^t}$ the *uniform* closure of its restriction to $\beta(x,b_0)$, composition with any ψ as above carries $\overline{\mathcal{B}^\epsilon}$ into $\overline{\mathcal{B}^M}$.

Let $\mathcal{F}$ and $\mathcal{G}$ be disjoint compact subsets of $\beta(x,b_0)\cap\tilde{Y}$ that do not meet $\overline{\mathbb{N}\times\{x\}}$. We will prove that there is (b_n) as in Lemma 1, whence the existence of K will follow.

Given $\xi\in\mathcal{F}$ and $\eta\in\mathcal{G}$ there is, by the separation assumption on $B|Y$, $(\alpha_n)\in\mathcal{B}^\varepsilon$ having opposite signs at ξ and η. The function $\varphi_1\circ(\alpha_n)\in\overline{\mathcal{B}^M}$ is positive at both points, so a linear combination of $\varphi_1\circ(\alpha_n)$ and (α_n) yields $(\gamma_n)\in\overline{\mathcal{B}^\varepsilon}$ such that $(\gamma_n)(\xi)>0$ and $(\gamma_n)(\eta)=0$. Then $(\lambda_n)=\varphi_1\circ(\gamma_n)\in\overline{\mathcal{B}^M}$ is non-negative, $(\lambda_n)(\xi)>0$, and $(\lambda_n)(\eta)=0$. It follows that $(\lambda_n)>0$ on some open neighbourhood of ξ. Using compactness of $\mathcal{F}$, adding the elements (λ_n) corresponding to finitely many points ξ gives (μ_n) in the uniform closure on $\beta(x,b_0)$ of some ball of $\ell^\infty(B_x)$ such that $(\mu_n)>0$ on $\mathcal{F}$ and $(\mu_n)(\eta)=0$. Adding a small multiple of (α_n) and then approximating uniformly on $\beta(x,b_0)$, we obtain $(\nu_n)\in\ell^\infty(B_x)$ such that $(\nu_n)>0$ on $\mathcal{F}$ and $(\nu_n)(\eta)<0$. If $t>0$ is sufficiently small, $(t\nu_n)\in\mathcal{B}^\varepsilon$ and $(\sigma_n)=(\varphi_1\circ\varphi_2)\circ(t\nu_n)\in\overline{\mathcal{B}^M}$ satisfies $(\sigma_n)\geqslant 0$ and $(\sigma_n)=0$ on $\mathcal{F}$; for a suitable choice of t, it will also be the case that $(\sigma_n)(\eta)>0$, and hence $(\sigma_n)>0$ on some open neighborhood of η. Adding the elements (σ_n) corresponding to finitely many points η gives (τ_n) in the uniform closure on $\beta(x,b_0)$ of some ball of $\ell^\infty(B_x)$ such that $(\tau_n)=0$ on $\mathcal{F}$ and $(\tau_n)>0$ on $\mathcal{G}$. Construct a similar element reversing the rôles of $\mathcal{F}$ and $\mathcal{G}$, take a linear combination of the two, and approximate from $\ell^\infty(B_x)$ uniformly on $\beta(x,b_0)$ to get the required (b_n). □

Proof of Theorem 1. Let $x\in X$ and let K be a compact neighbourhood of x such that $B|K=C(K)$ (provided by Lemma 2). Let $k>0$ be a number such that given $f\in C(K)$ there is $b\in B$ with $b=f$ on K and with $\|b\|\leqslant k\|f\|_{\infty,K}$, and let U be an open neighbourhood of x contained in K.

By the result of de Leeuw and Katznelson cited earlier [9], B is dense in $C(X)$, so we can find $b_0\in B$ satisfying $b_0(x)=0$, $b_0(X\backslash U)\subset(0,1)$ and $\|b_0\|<1$. Moreover, substracting from b_0 a small function in B_x that agrees with b_0 near x, we can assume that $b_0=0$ in an open neighbourhood V of x contained in U.

Let $\delta > 0$ be chosen small enough that $(b_0 + b)(X \backslash U) \subset (0, 1)$ and $\|b_0 + b\| < 1$ if $b \in B$ and $\|b\| < \delta$. Given $f \in C(X)$ with $f = 0$ on $X \backslash V$ and with $\|f\|_{\infty,K} < \delta/k$, we pick $b \in B$ with $b = f$ on K and $\|b\| < \delta$. Then $\varphi_2 \circ f = \varphi_2 \circ (b_0 + b) - \varphi_2 \circ b_0$, so we see that

$$\varphi_2 \circ f \in B$$

for any f in the open δ/k-ball of $C(X)_{X \backslash V}$ (the space of continuous functions that vanish on $X \backslash V$).

By the Baire category theorem there are a function $f_0 \in C(X)_{(X \backslash V) \cup \{x\}}$ and positive numbers ε, M such that $\|f_0 + f\|_\infty < \delta/k$ and $\varphi_2 \circ (f_0 + f) \in B \cap \overline{B(M)}$, where $\overline{B(M)}$ is the *uniform* closure on X of the M-ball of B, if f is in the ε-ball of $C(X)_{(X \backslash V) \cup \{x\}}$. If necessary perturbing f_0 slightly and shrinking ε, we can assume that $f_0 = 0$ in an open neighbourhood W of x contained in V. Then $\varphi_2 \circ f = \varphi_2 \circ (f_0 + f) - \varphi_2 \circ f_0$ if (in addition) $f = 0$ outside W so that

$$\varphi_2 \circ f \in B \cap \overline{B(2M)}$$

for all f in the ε-ball of $C(X)_{(X \backslash W) \cup \{x\}}$. Since φ_2 is not constant on any neighbourhood of 0 it follows that there is a positive number γ such that given any pair F, G of disjoint compact subsets of $W \backslash \{x\}$ there is $b \in B_x \cap \overline{B(2M)}$ with $b = 0$ outside W such that $b = \gamma$ on F, $b = 0$ on G. Standard approximation arguments now show that if $f \in C(X)_{X \backslash W}$ and $f(x) = 0$ then $f \in B$.

Finitely many of the neighbourhoods W cover X. Taking a corresponding partition of unity we find that there are finitely many points $x_1, \ldots, x_m \in X$ such that any $f \in C(X)$ that vanishes at these finitely many points is in B. Since, by the result of de Leeuw and Katznelson, there is $b \in B$ taking arbitrary values on a given finite set, we conclude that $B = C(X)$. □

Corollary 1. *Let B be a Banach function space on a compact Hausdorff space X that satisfies one (and hence all) of the local separation conditions at every $x \in X$. If B has a continuous non-affine operating function then $B = C(X)$.*

The following is due to Bernard [1], Sidney [10] and Hatori [5]:

Corollary 2. *Let $B = \Re(A)$ where A is a uniform algebra on X, and suppose that B has a non-affine operating function φ on some interval. Then $B = C(X)$ and $A = C_{\mathbb{C}}(X)$.*

To prove Corollary 2, we need a lemma often cited as "well known" without attribution or justification, for example in [10]. Inasmuch as we are not aware of any published proof of the full result, we are including one here.

Lemma 3. *With hypotheses as in the corollary, if X is infinite then φ must necessarily be continuous.*

In fact, K. Jarosz and Z. Sawoń [8] have proven continuity at *interior* points (relative to $\mathbb{R}$) of the domain of φ, though our proof is independent of this result.

Proof of Lemma 3. We will use a result to be found in [3, Part II, Proposition 10]; it is a generalization of a result of W. Spraglin [11, Theorem 3.1.8]. The result is this: If A is a uniform algebra on an infinite compact Hausdorff space X and if the complex-valued function F defined on some closed disc Δ operates from A into $C_{\mathbb{C}}(X)$ in the sense that $F \circ f \in C_{\mathbb{C}}(X)$ whenever $f \in A$ has range contained in Δ, then F must be continuous.

Let I be any non-degenerate closed interval contained in the domain of φ, and let Δ be the closed disc with diameter I. Define $F : \Delta \to \mathbb{R}$ by $F(z) = \varphi(\Re(z))$ and apply the quoted result to obtain that F is continuous on Δ, hence the restriction of φ to I is continuous. □

Note that φ need only operate from $\Re(A)$ into $C(X)$ to deduce that φ is continuous.

Proof of Corollary 2. If X is finite, then $A = C_{\mathbb{C}}(X)$ is trivial, so $B = C(X)$. When X is infinite, Lemma 3 shows that φ is continuous. By the de Leeuw–Katznelson theorem [9], B is dense in $C(X)$, so A is a Dirichlet algebra. From Example 2 it follows that B satisfies the local separation condition at every point of X, so by Corollary 1, $B = C(X)$. Now $A = C_{\mathbb{C}}(X)$ by, for instance, Wermer's theorem [12]. □

The local separation condition in Corollary 1 cannot be replaced by the weaker global separation condition, as the following example due to Hatori [7] shows.

Example 3. Let X and B be as in Example 1, and let $\ell_1(X_+)$ be the space

$$\ell_1(X_+) = \left\{ f \in C(X)\colon\ f(x) = 0 \text{ for } x \leqslant 0 \text{ and } \|f\|_1 := \sum_{n=1}^{\infty} \left| f(1/n) \right| < \infty \right\}.$$

Clearly $B \cap \ell_1(X_+) = \{0\}$, and $B_1 = B \oplus \ell_1(X_+)$ is a Banach function space on X with the norm $\|b + f\| = \|b\|_\infty + \|f\|_1$ for $b \in B$ and $f \in \ell_1(X_+)$. Alternatively, B_1 consists of those continuous functions g such that $\sum_{n=1}^{\infty} |g(1/n) - (1/2)g(-1/n)| < \infty$. Since B satisfies the global separation condition, so does B_1. Because $||g(1/n)| - (1/2)|g(-1/n)|| \leqslant |g(1/n) - (1/2)g(-1/n)|$, the non-affine continuous function $\varphi(t) = |t|$ operates on B_1.

4. Fibers

There is a natural decomposition of $\tilde{X}$ into fibers that are finer than $\beta(x, f)$ and can often be used to prove variants of the key Lemma 1, Bernard's lemma. Let X be a compact Hausdorff space, and for $x \in X$ consider the fiber over x in $\tilde{X}$

$$F_x = \bigcap \overline{\mathbb{N} \times K},$$

where K varies over all compact neighbourhoods of x (or any base of compact neighbourhoods of x will suffice). Alternative descriptions are

$$F_x = \bigcap \{\beta(x, f):\ f \in C(X)\} = \{\xi \in \tilde{X}:\ (f)(\xi) = f(x)\ \forall f \in C(X)\}.$$

If B is any point-separating subset of $C(X)$, it suffices in these descriptions to take just those f that belong to B. $\overline{\mathbb{N} \times \{x\}} \subset F_x$, and $\tilde{X}$ is the disjoint union of the sets F_x as x varies over X.

We now verify that, in general, F_x cannot replace $\beta(x, f)$ in Lemma 1.

Example 4. Let $Y = [0, \Omega]$, the space of all ordinal numbers ω not exceeding the first uncountable ordinal number Ω, in the interval topology; thus a subbase for the topology of Y is given by the sets $[0, \gamma) = \{\omega:\ 0 \leqslant \omega < \gamma\}$ for $0 < \gamma \in Y$ together with the sets $(\gamma, \Omega] = \{\omega:\ \gamma < \omega \leqslant \Omega\}$ for $\Omega > \gamma \in Y$. Y is a compact Hausdorff space, and given any countable set of continuous real-valued function on Y there is a neighbourhood of Ω on which *all* the functions in the set are constant.

Let $\mathbb{D}$ consist of the complex numbers of modulus < 1, and denote by $\overline{\mathbb{D}}$ its closure, the set of complex numbers of modulus $\leqslant 1$. Let $X_1 = Y \times \overline{\mathbb{D}}$ and let B_1 consist of the continuous functions b on X_1 such that $b(\omega, \cdot)$ is harmonic on $\mathbb{D}$ for every $\omega \in Y$, and $b(\Omega, \cdot)$ is constant on $\overline{\mathbb{D}}$. B_1 may be regarded as a Banach function space B on X, the quotient space obtained from X_1 by collapsing the set $\{\Omega\} \times \overline{\mathbb{D}}$ to a point x_Ω; the norm on B is the uniform norm. Given any countable subset of $C(X)$, the point x_Ω has a neighbourhood in X on which every function in the set is constant.

Suppose $\tilde{f} = (f_n)$ belongs to $\widetilde{C(X)} = C(\tilde{X})$. Let K be a compact neighbourhood of x_Ω on which every f_n is constant. Let c_n denote the constant function on X that agrees with f_n on K, and $\tilde{c} = (c_n)$. Then $\tilde{f} = \tilde{c}$ on $\mathbb{N} \times K$, so on its closure in $\tilde{X}$, and so on F_{x_Ω}. It follows (since $\overline{\mathbb{N} \times \{x_\Omega\}} = \beta(\mathbb{N} \times \{x_\Omega\})$ is the set of nonzero real-valued homomorphisms of the real Banach algebra ℓ^∞) that $F_{x_\Omega} = \overline{\mathbb{N} \times \{x_\Omega\}}$ and that the restriction of $\tilde{B}$ to F_{x_Ω} consists of all continuous functions on the latter. Consequently, the main hypothesis of Lemma 1, with F_x in place of $\beta(x, f)$, is vacuously satisfied for $x = x_\Omega$.

On the other hand, there is no compact neighbourhood K of x_Ω such that $B|K = C(K)$, or even such that $B|K$ is ultraseparating on K. For K must contain $\{\omega\} \times \overline{\mathbb{D}}$ for some (in fact, many) $\omega < \Omega$, and we fix one such ω. Standard inequalities for harmonic functions show that if $b \in B$ then $|b(\omega, z) - b(\omega, 0)| \leqslant 12\|b\| \cdot |z|$ if $z \in \mathbb{D}$ and $|z| \leqslant 1/2$. Therefore, for any r, $0 < r \leqslant 1/2$, if $b \in B$ and $b > 1$ on $\mathcal{F} = \{(\omega, r)\}$, $b < -1$ on $\mathcal{G} = \{(\omega, 0)\}$, we have $\|b\| > 1/(6r)$. Since r can be taken arbitrarily small, no M works (for any N) in the second version of the global separation condition for $B|K$.

It turns out that Lemma 1 holds with F_x in place of $\beta(x, f)$ provided the one-point set $\{x\}$ is a G_δ-set, which is automatically the case if X is a metric space. In fact, there are extensions valid in complete generality, if we replace $\tilde{X}$ by $\tilde{X}^\Lambda = \beta(\Lambda \times X)$ where Λ is an infinite discrete space of appropriate cardinality. $\ell^\infty(\Lambda, B)$, the space of bounded B-valued functions on Λ, can be interpreted as a Banach function space $\tilde{B}^\Lambda$ on $\tilde{X}^\Lambda$, and the fiber over x in $\tilde{X}^\Lambda$ is $F_x^\Lambda = \bigcap \overline{\Lambda \times K}$, the intersection taken as K runs through any base for the topology of X at x consisting of compact neighbourhoods of x. Properties of the F_x^Λ are investigated systematically in [6] and [7].

The proof of Lemma 1 used the fact that $\beta(x, f)$ was the intersection of a family of open sets in $\tilde{X}$ that could be mapped injectively into the index set $\mathbb{N}$ used to construct $\tilde{X}$. A corresponding family of open sets for F_x^Λ is given by a base for the topology of X at x. Therefore we will require that Λ have cardinality at least that of such a base.

In Theorem 12 in [7] the conclusion of Theorem 1 is obtained using the following separation condition:

for every $x \in X$ and every pair of different points p and q in $F_x^\Lambda \setminus \overline{\Lambda \times \{x\}}$ there is a function $\tilde{f} \in \widetilde{B_x}^\Lambda$ with $\tilde{f}(p) = 1$ and $\tilde{f}(q) = 0$.

This separation condition clearly implies that $lat(\widetilde{B_x}^\Lambda)|F_x^\Lambda$ is dense in $C(F_x^\Lambda)_{\overline{\Lambda \times \{x\}}}$. Note that the converse does not hold. Thus the next theorem shows that Theorem 1 includes Theorem 12 in [7].

Theorem 2. *Let B be Banach function space on X, let $x \in X$, and let Λ be an infinite discrete space of cardinality at least that of a base for the topology of X at x. If $lat(\widetilde{B_x}^\Lambda)|F_x^\Lambda$ is dense in $C(F_x^\Lambda)_{\overline{\Lambda \times \{x\}}}$, then $B|K$ satisfies the local separation conditions at x for some compact neighbourhood K of x.*

Proof. Assume that the hypotheses of the proposition are true but the conclusion is false. Let $\{K_\gamma\colon\ \gamma \in \Gamma\}$ be a family of compact neighbourhoods of x that forms a base for the topology of X at x and has cardinality no greater than that of Λ. Given $\gamma \in \Gamma$, there can be no M and N as in the first local separation condition for $B|K_\gamma$ at x, so for every $n \in \mathbb{N}$ there are disjoint compact subsets $F_{\gamma,n}$, $G_{\gamma,n}$ of $K_\gamma \backslash \{x\}$ such that for $b_i, c_i \in B_x$, the inequalities $\|b_i\|, \|c_i\| \leqslant n$ $(1 \leqslant i \leqslant n)$, $b_1 \wedge \cdots \wedge b_n - c_1 \wedge \cdots \wedge c_n > 1$ on $F_{\gamma,n}$ and $b_1 \wedge \cdots \wedge b_n - c_1 \wedge \cdots \wedge c_n < 0$ on $G_{\gamma,n}$ are incompatible.

If necessary replacing Λ by another set of the same cardinality, we may assume that $\Lambda = (\Gamma \times \mathbb{N}) \cup \Theta$ disjointly for some set Θ. For $(\gamma, n) \in \Gamma \times \mathbb{N}$ let $f_{(\gamma,n)} \in C(X)_x$ satisfy $\|f_{(\gamma,n)}\|_\infty = 2$, $f_{(\gamma,n)} \equiv 2$ on $F_{\gamma,n}$, and $f_{(\gamma,n)} \equiv -1$ on $G_{\gamma,n}$, and for $\theta \in \Theta$ let $f_\theta \equiv 0$ on X. $(f_\lambda)_{\lambda \in \Lambda}$ gives an element $\tilde{f}$ of $\widetilde{C(X)_x}^{\Lambda} = C(\tilde{X}^\Lambda)_{\overline{\Lambda \times \{x\}}}$. The hypotheses show that $lat(\widetilde{B_x}^{\Lambda})|F_x^\Lambda$ is dense in $C(F_x^\Lambda)_{\overline{\Lambda \times \{x\}}}$, so there is $\tilde{u} = (u_\lambda)_{\lambda \in \Lambda} \in lat(\widetilde{B_x}^{\Lambda})$ such that $|\tilde{u} - \tilde{f}| < 1$ on F_x^Λ, and this inequality persists on an open neighbourhood U of F_x^Λ in $\tilde{X}^\Lambda$. $\Lambda \times K_{\gamma_0} \subset U$ for some $\gamma_0 \in \Gamma$, and $\tilde{u}$ can be written as $\tilde{u} = \widetilde{b_1} \wedge \cdots \wedge \widetilde{b_{n_0}} - \widetilde{c_1} \wedge \cdots \wedge \widetilde{c_{n_0}}$ for some $n_0 \in \mathbb{N}$ and elements $\widetilde{b_i} = (b_{i,\lambda})_{\lambda \in \Lambda}$ and $\widetilde{c_i} = (c_{i,\lambda})_{\lambda \in \Lambda}$ of $\widetilde{B_x}^{\Lambda}$.

Choose $n_1 \in \mathbb{N}$ so that $n_1 \geqslant n_0$ and $n_1 \geqslant \|\widetilde{b_1}\| \vee \cdots \vee \|\widetilde{b_{n_0}}\| \vee \|\widetilde{c_1}\| \vee \cdots \vee \|\widetilde{c_{n_0}}\|$. Repeating some of the $\widetilde{b_i}$ and $\widetilde{c_i}$, we can rewrite $\tilde{u}$ as $\tilde{u} = \widetilde{b_1} \wedge \cdots \wedge \widetilde{b_{n_1}} - \widetilde{c_1} \wedge \cdots \wedge \widetilde{c_{n_1}}$. The inequality $|\tilde{u} - \tilde{f}| < 1$ holds on $\{(\gamma_0, n_1)\} \times K_{\gamma_0}$, that is, $|u_{(\gamma_0,n_1)} - f_{(\gamma_0,n_1)}| < 1$ on K_{γ_0}. It follows that $u_{(\gamma_0,n_1)} > 1$ on F_{γ_0,n_1} and $u_{(\gamma_0,n_1)} < 0$ on G_{γ_0,n_1}, which with the representation $u_{(\gamma_0,n_1)} = b_{1,(\gamma_0,n_1)} \wedge \cdots \wedge b_{n_1,(\gamma_0,n_1)} - c_{1,(\gamma_0,n_1)} \wedge \cdots \wedge c_{n_1,(\gamma_0,n_1)}$ and the inequalities $\|b_{i,(\gamma_0,n_1)}\|, \|c_{i,(\gamma_0,n_1)}\| \leqslant n_1$ gives a contradiction to the manner in which F_{γ_0,n_1} and G_{γ_0,n_1} were chosen. □

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