Direct Limits of Commutative Spaces

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G loc cpt group, K cpt subgroup. Equivalent:

(1) (G,K) commutative, i.e. (G,K) Gelfand pair, i.e. $L^1(K\backslash G/K)$ commutative

(2) $C_c(K\backslash G/K)$ is commutative

(3) $g, g' \in G \Rightarrow \mu_{KgK} * \mu_{Kg'K} = \mu_{Kg'K} * \mu_{KgK}$

(4) measure algebra $\mathcal{M}(K\backslash G/K)$ commutative

(5) rep of G on $L^2(G/K)$ multiplicity free

And if G is a connected Lie group

(6) G–invariant diff op on G/K commute

For G, K infinite dimensional (5) still makes sense if we have an appropriate space for $L^2(G/K)$

Def. Let $G = \varinjlim G_n$ limit group, $\pi = \varinjlim \pi_n$ limit rep, $V = \varinjlim V_n$ rep space. Suppose each V_n is $\oplus$ of primaries $U_{i,n}$ and each $U_{i,n}$ contained in some $U_{j,n+1}$. Then π is **limit–aligned**.

Theorem. A limit–aligned rep $\pi = \varinjlim \pi_n$ is $\oplus$ of primaries. If the π_n are multiplicity free then π is a multiplicity free direct sum of irreducible representations.

Def. π is **lim–irreducible** if it is irreducible and has form $\pi = \varinjlim \pi_n$ with π_n irreducible.

Theorem. Let $\pi = \varinjlim \pi_n$, each π_n multiplicity free direct sum of irreducible highest weight reps. Suppose for $n >> 0$ that $V_{n-1} \hookrightarrow V_n$ sends highest weight vector to highest weight vector. Then π is a multiplicity free $\oplus$ of lim–irreducible representations.

Compact Irreducible Riemannian Symmetric $M_n = G_n/K_n$				
	G_n	K_n	RankM_n	DimM_n
1	$SU(n)\times SU(n)$	diagonal $SU(n)$	$n-1$	n^2-1
2	$Spin(2n+1)\times Spin(2n+1)$	diagonal $Spin(2n+1)$	n	$2n^2+n$
3	$Spin(2n)\times Spin(2n)$	diagonal $Spin(2n)$	n	$2n^2-n$
4	$Sp(n)\times Sp(n)$	diagonal $Sp(n)$	n	$2n^2+n$
5	$SU(p+q),\ p=p_n, q=q_n$	$S(U(p)\times U(q))$	$\min(p,q)$	$2pq$
6	$SU(n)$	$SO(n)$	$n-1$	$\frac{(n-1)(n+2)}{2}$
7	$SU(2n)$	$Sp(n)$	$n-1$	$2n^2-n-1$
8	$SO(p+q),\ p=p_n, q=q_n$	$SO(p)\times SO(q)$	$\min(p,q)$	pq
9	$SO(2n)$	$U(n)$	$[\frac{n}{2}]$	$n(n-1)$
10	$Sp(p+q),\ p=p_n, q=q_n$	$Sp(p)\times Sp(q)$	$\min(p,q)$	$4pq$
11	$Sp(n)$	$U(n)$	n	$n(n+1)$

$\mathfrak{g}_n = \mathfrak{k}_n + \mathfrak{s}_n$ and $\mathfrak{a}_n \subset \mathfrak{s}_n$ max abelian

$$\mathfrak{k}_n \subset \mathfrak{k}_{n+1}, \quad \mathfrak{s}_n \subset \mathfrak{s}_{n+1}, \quad \mathfrak{a}_n \subset \mathfrak{a}_{n+1}$$

$\Sigma_n = \Sigma_n(\mathfrak{g}_n, \mathfrak{a}_n)$ and $\Sigma = \varprojlim \Sigma_n$ root systems

$\Psi_n = \Psi_n(\mathfrak{g}_n, \mathfrak{a}_n) = \{\psi_{1,n}, \dots, \psi_{r_n,n}\}$ simple $\mathfrak{a}_n$–root system for $\mathfrak{g}_n$

Case by case cofinal: if $\psi_{i,n} \in \Psi_n$ there is unique $\psi_{j,n+1} \in \Psi_{n+1}$ with $\psi_{j,n+1}|_{\mathfrak{a}_n} = \psi_{i,n}$

Now assume the Ψ_n ordered: $\psi_{i,n} = \psi_{i,n+1}|_{\mathfrak{a}_n}$

$\Xi_n = \{\xi_{1,n}, \dots, \xi_{r_n,n}\}$ fund highest weights

$\pi_{I,n}$ rep of G_n high weight $\xi_I = i_1\xi_1 + \cdots + i_p\xi_{r_n}$

Cartan–Helgason: $L^2(G_n/K_n) = \sum_I V_{I,n}$

if $u, v \in V_{I,n}$ then $f_{u,v;I,n}$ matrix coef

$\langle f_{u,v;I,n}, f_{u',v';I,n}\rangle|_{L^2(G_n)} =$
$(\deg \pi_{I,n})^{-1}\langle u, u'\rangle\overline{\langle v, v'\rangle}$ (Schur Orthog)

$v_{I,n} \in V_{I,n} \subset L^2(G_n/K_n)$: high wt unit vector

$V_{I,n} \hookrightarrow V_{I,n+1}$ by $V_{I,n} \cong \mathcal{U}(\mathfrak{g}_n)(v_{n+1})|_{G_n/K_n}$

$f \mapsto (\deg \pi_{I,n+1}/\deg \pi_{I,n})^{1/2} f$ is a unitary injection of $V_{I,n}$ into $V_{I,n+1}$, defines unitary injection $L^2(G_n/K_n) \hookrightarrow L^2(G_{n+1}/K_{n+1})$.

Def $L^2(G/K) = \varinjlim L^2(G_n/K_n)$.

Theorem The left regular representation of G on $L^2(G/K)$ is a multiplicity free discrete direct sum of lim–irreducible representations $\pi_I = \varinjlim \pi_{I,n}$. This applies to all the direct systems of the symmetric space table above.

This gives thirteen infinite dimensional multiplicity free spaces

1. $SU(\infty) \times SU(\infty)/\mathrm{d}iag\, SU(\infty),$ group manifold $SU(\infty)$,
2. $Spin(\infty) \times Spin(\infty)/\mathrm{d}iag\, Spin(\infty),$ group manifold $Spin(\infty)$,
3. $Sp(\infty) \times Sp(\infty)/\mathrm{d}iag\, Sp(\infty),$ group manifold $Sp(\infty)$,
4. $SU(p+\infty)/S(U(p) \times U(\infty)),$ $\mathbb{C}^p$ subspaces of $\mathbb{C}^\infty$,
5. $SU(2\infty)/[S(U(\infty) \times U(\infty))],$ $\mathbb{C}^\infty$ subspaces of infinite codim in $\mathbb{C}^\infty$,
6. $SU(\infty)/SO(\infty)$,
7. $SU(2\infty)/Sp(\infty)$,
8. $SO(p+\infty)/[SO(p) \times SO(\infty)],$ oriented $\mathbb{R}^p$ subspaces of $\mathbb{R}^\infty$,
9. $SO(2\infty)/[SO(\infty) \times SO(\infty)],$ $\mathbb{R}^\infty$ subspaces of infinite codim in $\mathbb{R}^\infty$,
10. $SO(2\infty)/U(\infty)$,
11. $Sp(p+\infty)/[Sp(p) \times Sp(\infty)],$ $\mathbb{H}^p$ subspaces of $\mathbb{H}^\infty$,
12. $Sp(2\infty)/[Sp(\infty) \times Sp(\infty)],$ $\mathbb{H}^\infty$ subspaces of infinite codim in $\mathbb{H}^\infty$,
13. $Sp(\infty)/U(\infty)$.

$M_n = G_n/H_n$ **Weakly Symmetric**				G_n/K_n **Symmetric**
	G_n	H_n	conditions	K_n with $H_n \subset K_n \subset G_n$
1	$SU(m+n)$	$SU(m) \times SU(n)]$	$n > m \geqq 1$	$S[U(m) \times U(n)]$
2	$SO(2n)$	$SU(n)$	n odd, $n \geqq 3$	$U(n)$
3	E_6	$Spin(10)$		$Spin(10) \cdot Spin(2)$
4	$SU(2n+1)$	$Sp(n)$	$n \geqq 1$	$U(2n) = S[U(2n) \times U(1)]$
5	$SU(2n+1)$	$Sp(n) \times U(1)$	$n \geqq 1$	$U(2n) = S[U(2n) \times U(1)]$
6	$Spin(7)$	G_2		(there is none)
7	G_2	$SU(3)$		(there is none)
8	$SO(10)$	$Spin(7) \times SO(2)$		$SO(8) \times SO(2)$
9	$SO(9)$	$Spin(7)$		$SO(8)$
10	$Spin(8)$	G_2		$Spin(7)$
11	$SO(2n+1)$	$U(n)$	$n \geqq 2$	$SO(2n)$
12	$Sp(n)$	$Sp(n-1) \times U(1)$	$n \geqq 1$	$Sp(n-1) \times Sp(1)$

That gives us the nonsymmetric direct systems $\{(G_n, K_n)\}$ where

(a) $G_n = SU(p_n + q_n)$ and $K_n = SU(p_n) \times SU(q_n),\ p_n < q_n$
(b) $G_n = SO(2n)$ and $K_n = SU(n),\ n$ odd, $n \geqq 3$
(c) $G_n = SU(2n+1)$ and $K_n = Sp(n),\ n \geqq 1$
(d) $G_n = SU(2n+1)$ and $K_n = U(1) \times Sp(n),\ n \geqq 1$
(e) $G_n = SO(2n+1)$ and $K_n = U(n),\ n \geqq 2$
(f) $G_n = Sp(n)$ and $K_n = U(1) \times Sp(n-1),\ n \geqq 2$

Def. Let $\pi = \varinjlim \pi_n$ lim–irreducible rep of $G = \varinjlim G_n$, space $V = \varinjlim V_n$, such that (i) $\pi_n(K_n)$ is the $\pi_n(G_n)$–stabilizer of $v_n \in V_n$ and (ii) $v_n = v_{n-1} + w_n$ where $\pi_n(G_n)$ leaves w_n fixed. ($\{v_n\}$ gives a coherent system of embeddings of the G_n/K_n.) If for $n >> 0$ the π_n have the same high wt vector, then $\pi = \varinjlim \pi_n$ is a **defining representation** for $\{(G_n, K_n)\}$.

The systems (a) with $\{p_n\}$ bounded, (d), (e) and (f) have defining representations.

Given defining rep $\pi = \varinjlim \pi_n$ for $\{(G_n, K_n)\}$:

$\mathcal{A}(G_n)$: all $\mathbb{C}$-valued $\mathbb{R}$–poly functions on V_n

$\mathcal{A}(G_n) \hookrightarrow \mathcal{A}(G_{n+1})$ by "ignore new variables"

$\mathcal{A}(G_n/K_n) \hookrightarrow \mathcal{A}(G_{n+1}/K_{n+1})$ well defined

$\mathcal{A}(G/K) := \varinjlim \mathcal{A}(G_n/K_n)$, G–module

$\mathcal{C}(G_n/K_n)$: uniform limit completion of $\mathcal{A}(G_n/K_n)$

$\mathcal{C}(G/K) := \varinjlim \mathcal{C}(G_n/K_n)$, G–module

For L^2 we need $f \mapsto (\deg \pi_{I,n+1} / \deg \pi_{I,n})^{1/2} f$ of inner products due to Schur Orthogonality

Def. $\{(G_n, K_n)\}$ is **parabolic** if $(G_n)_{\mathbb{C}}$ is the ss part of a parabolic subgroup of $(G_{n+1})_{\mathbb{C}}$. In that case we can align the fundamental simple weights as in the symmetric spaces case, and if $V_n \hookrightarrow V_{n+1}$ maps K_n–invariants to K_{n+1}–invariants we have equivariant unitary injections $L^2(G_n/K_n) \hookrightarrow L^2(G_{n+1}/K_{n+1})$

Theorem Let $\{(G_n, K_n)\}$ parabolic system of commutative pairs with defining representation, where $V_n \hookrightarrow V_{n+1}$ maps K_n–invariants to K_{n+1}–invariants. Then the regular reps of G on $\mathcal{A}(G/K), \mathcal{C}(G/K)$ and $L^2(G/K)$ are multiplicity free direct sums of lim–irred representations.

This theorem applies to the systems (a) with $\{p_n\}$ bounded, (d), (e) and (f) above

Now consider the direct systems $\{(G_n, K_n)\}$ of commutative pairs given by

(1) $G_n = SO(n_0 + 2n)$, $K_n = SO(n_0 + 2n - 1)$,

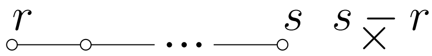
(2) $G_n = SU(p + n)$, $K_n = SU(p) \times SU(n)$,

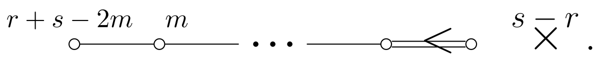
(3) $G_n = Sp(n) \times Sp(1)$, $K_n = Sp(n-1) \times Sp(1)_{diag}$,

(4) $G_n = Sp(n) \times U(1)$, $K_n = Sp(n-1) \times U(1)_{diag}$,

(5) $G_n = Sp(n) \times Sp(1)$, $K_n = Sp(n-1) \times U(1)_{diag}$,

(6) $G_n = Sp(n)$, $K_n = Sp(n-1) \times U(1)$.

The multiplicity free and lim–aligned properties for (1) are immediate from classical facts about the action ψ_{ℓ, n_0+2n} of $SO(n_0 + 2n)$ on the polynomials of degree ℓ on $\mathbb{R}^{n_0+2n}$. They hold for (2) using the rep $\psi_{r,s;n}$ of G on space of polynomials of degree r in the z_i, s in the $\overline{z_j}$, with the branching $\psi_{\ell,2n}|_{U(n)} = \sum_{r+s=\ell} \psi_{r,s;n}$. Here $\psi_{r,s;n}$ has diagram 

. The action in the four quaternion cases comes from $\psi_{r,s;2n}|_{Sp(n)U(1)} = \sum_{0 \leqq m \leqq \min(r,s)} \varphi_{r,s,m;n}$ where $\varphi_{r,s,m;n}$ is the representation of $Sp(n)U(1)$ whose Dynkin diagram is 

. So we have a some infinite dimensional spaces, related to spheres and Grassmann manifolds, with the multiplicity free properties.

H_n: $2n+1$–dim Heisenberg group $\operatorname{Im}\mathbb{C} + \mathbb{C}^n$

K_n: closed subgroup of $U(n)(\subset \operatorname{Aut}(H_n))$

Direct systems $\{(H_n \rtimes K_n, K_n)\}$ of Gelfand pairs, K_n connected and irreducible on $\mathbb{C}^n$			
	Group K_n	Acting on	Conditions on n
1	$SU(n)$	$\mathbb{C}^n$	$n \geqq 2$
2	$U(n)$	$\mathbb{C}^n$	$n \geqq 1$
3	$Sp(m)$	$\mathbb{C}^n$	$n = 2m$
4	$U(1) \times Sp(m)$	$\mathbb{C}^n$	$n = 2m$
5a	$U(1) \times SO(2m)$	$\mathbb{C}^{2m}$	$n = 2m \geqq 2$
5b	$U(1) \times SO(2m+1)$	$\mathbb{C}^{2m+1}$	$n = 2m+1 \geqq 3$
6	$U(m)$	$S^2(\mathbb{C}^m)$	$m \geqq 2,\ n = \frac{1}{2}m(m+1)$
7	$SU(m)$	$\Lambda^2(\mathbb{C}^m)$	m odd, $n = \frac{1}{2}m(m-1)$
8	$U(m)$	$\Lambda^2(\mathbb{C}^m)$	$n = \frac{1}{2}m(m-1)$
9	$SU(\ell) \times SU(m)$	$\mathbb{C}^\ell \otimes \mathbb{C}^m$	$n = \ell m,\ \ell \neq m$
10	$S(U(\ell) \times U(m))$	$\mathbb{C}^\ell \otimes \mathbb{C}^m$	$n = \ell m$
11	$U(2) \times Sp(m)$	$\mathbb{C}^2 \otimes \mathbb{C}^{2m}$	$n = 4m$
12	$SU(3) \times Sp(m)$	$\mathbb{C}^3 \otimes \mathbb{C}^{2m}$	$n = 6m$
13	$U(3) \times Sp(m)$	$\mathbb{C}^3 \otimes \mathbb{C}^{2m}$	$n = 6m$
15	$SU(m) \times Sp(4)$	$\mathbb{C}^m \otimes \mathbb{C}^8$	$n = 8m,\ m \geqq 3$
16	$U(m) \times Sp(4)$	$\mathbb{C}^m \otimes \mathbb{C}^8$	$n = 8m,\ m \geqq 3$

Extend the notion of lim–aligned to allow continuous direct sums of representations

Theorem. In each of the 16 systems $\{(G_n, K_n)\}$, $G_n = H_n \rtimes K_n$, just listed,

- $\{K_n\}$ is strict and parabolic,
- $\{L^2(H_n \rtimes K_n)\}$ forms a strict direct system, unitary injections scaled using Schur Orthog,
- if $r > n$ then every K_n–invariant element of $L^2(H_n \rtimes K_n)$ is the image of a K_r–invariant element of $L^2(H_r \rtimes K_r)$ under the adjoint of the inclusion map
- $\{L^2(G_n/K_n)\}$ strict unitary direct system
- let $G = \varinjlim G_n$, $K = \varinjlim K_n$, $L^2(G/K) = \varinjlim L^2(G_n/K_n)$; then the regular representation of G on $L^2(G/K)$ is a multiplicity free direct integral of lim–aligned irreducible representations.