Short Course on Commutative Spaces

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Lecture 1, August 8: Basics

(G, K) is a **Gelfand pair** if the convolution algebra $L^1(K\backslash G/K)$ is commutative.

If G has an involutive automorphism θ such that $\theta(g^{-1}) \in KgK$ for all $g \in G$ then (G, K) is a Gelfand pair.

If there is an involutive automorphism θ of G such that $\theta(K) = K$ and $G = KP$ where $\theta(p) = p^{-1}$ for all $p \in P$ then (G, K) is a Gelfand pair.

If (G, K) Gelfand pair then G is unimodular.

Examples:

If $X = G/K$ is a riemannian symmetric space and $G = \mathcal{I}(X)^0$ then (G, K) is a Gelfand pair.

More generally, if $X = G/K$ is a weakly symmetric riemannian manifold and $G = \mathcal{I}(X)^0$ then (G, K) is a Gelfand pair.

If G is a locally compact abelian group then $(G, \{1\})$ is a Gelfand pair.

If K is a compact group then $(K \times K, \delta K)$ is a Gelfand pair.

If Γ is a homogeneous tree, $\Gamma = G/K$ where $G = \mathrm{Aut}(\Gamma)$, then (G, K) is a Gelfand pair.

If $\mathfrak{K}$ is a field complete rel discrete valuation, $\mathfrak{O}$ ring of integers in $\mathfrak{K}$, then $(PGL(2, \mathfrak{K}), PGL(2, \mathfrak{O}))$ is a Gelfand pair.

Equivalent Conditions:

1. (G, K) is a Gelfand pair.

2. If $g, g' \in G$ then

$$(KgK)(Kg'K) = (Kg'K)(KgK)$$

multiplication of sets.

3. If $g, g' \in G$ then

$$\mu_{KgK} * \mu_{Kg'K} = \mu_{Kg'K} * \mu_{KgK}$$

convolution of measures.

4. (For the case where G is a connected Lie group.) The algebra $\mathcal{D}(G, K)$ of G–invariant differential operators on G/K is commutative.

Spherical Measures, Spherical Functions.

Spherical measure for (G, K): nonzero K–bi–invariant Radon measure m on G such that

$$f \mapsto m(f) = \int_G f(g)\, dm(g)$$

algebra homomorphism $C_c(K\backslash G/K) \to \mathbb{C}$.

Spherical measures for (G, K) are absolutely continuous with respect to Haar measure on G.

Spherical function for (G, K): continuous $\omega : G \to \mathbb{C}$ such that the measure

$$m(f) = \int_G f(x)\omega(x^{-1})\, d\mu_G(x)$$

is spherical for (G, K). (It is automatic that ω is K–bi–invariant and that $\omega(1) = 1$.)

Equivalent:

(i) $\omega : G \longrightarrow \mathbb{C}$ is spherical for (G, K),

(ii) (Joint Eigenvalue) $\omega : G \longrightarrow \mathbb{C}$ is a continuous K–bi–invariant function with $\omega(1) = 1$ and such that if $f \in C_c(K\backslash G/K)$ there exists $\lambda_f \in \mathbb{C}$ with $f * \omega = \lambda_f \, \omega$,

(iii) (Functional Equation) $\omega : G \longrightarrow \mathbb{C}$ is a continuous function not identically zero; and if $g_1, g_2 \in G$ then $\omega(g_1)\omega(g_2) = \int_K \omega(g_1 k g_2)\, d\mu_K(k)$.

If G is an LCA group and $K = \{1\}$ then (iii) shows that the spherical functions for (G, K) are just the quasi–characters on G.

If G is compact then the spherical functions for (G, K) are just the $\omega_\kappa(g) = \int_K \operatorname{trace} \kappa(gk) d\mu_K(k)$ for $\kappa \in \widehat{G}$.

Positive Definite Functions.

$\phi : G \to \mathbb{C}$ is **positive definite** if

$$\sum_1^n \phi(g_j^{-1} g_i)\overline{c_j} c_i \geqq 0$$

for $\{g_1, \dots, g_n\} \subset G$ and $\{c_1, \dots, c_n\} \subset \mathbb{C}$.

If $\phi : G \to \mathbb{C}$ continuous pos def with $\phi(1) = 1$, then there exist a unitary representation π of G and a cyclic unit vector $u \in H_\pi$ such that $\phi(g) = \langle u, \pi(g)u \rangle$ for all $g \in G$. (π, u) unique: if (π', u') is another such pair then there is a unitary equivalence $A \in I(\pi, \pi')$ such that $A(u) = u'$. If ϕ is (G, K)–spherical then π is irreducible, $\pi(k)u = u$ for all $k \in K$, and u spans the space H_π^K of K–fixed vectors in H_π .

If $\pi \in \widehat{G}$, $\dim H_\pi^K = 1$, H_π^K spanned by unit vector u, then $\phi(g) = \langle u, \pi(g)u \rangle$ positive definite spherical and (π, u) corresponds to ϕ.

Example:

$G/K = \mathbb{E}^n$ euclidean space with

$K \subset O(n)$ closed, $G = \mathbb{R}^n \rtimes K$ semidirect

$\omega_\xi^K : \mathbb{E}^n \to \mathbb{C}$ by

$$\omega_\xi^K(xK) = \int_K \exp i\langle x, k\xi\rangle d\mu_K(k), \quad \xi \in \mathbb{C}^n$$

ω_ξ^K (up to G): the (G,K)–spherical functions

$\xi \in \mathbb{R}^n$ (ξ real) $\Rightarrow \omega_\xi^K$ positive definite

For $n > 1$, $\omega_\xi^{SO(n)}(xk) =$

$$\frac{\pi^{-1/2}\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\int_0^\pi \cos(\sqrt{b(\xi,\xi)}||x||\cos\theta)\sin^{n-2}\theta\, d\theta$$

Example:

G/K compact riemannian symmetric, rank 1:

X	m_γ	$m_{2\gamma}$	ρ	β
$S^q, P^q(\mathbb{R})$	$q-1$	0	$\frac{1}{2}(q-1)\gamma$	γ
$P^q(\mathbb{C})$	$2q-2$	1	$q\gamma$	2γ
$P^q(\mathbb{H})$	$4q-4$	3	$(2q+1)\gamma$	2γ
$P^2(\mathbb{O})$	8	7	11γ	2γ

γ simple restricted root, β max restricted root

$\xi_0 \in \mathfrak{a}$ defined by $\gamma(\xi_0) = i$

The $\pi_{n\beta} \in \widehat{G}$, max weight $n\beta, n = 0, 1, \dots$ are the K–spherical reps of G. Corresp spherical function $\omega_{n\beta}(\exp(t\xi_0)) =$

$${}_2F_1\left(\tfrac{1}{2}m_{\beta/2} + m_\beta + n, -n, \tfrac{1}{2}(m_{\beta/2} + m_\beta + 1); \sin^2\left(\tfrac{\langle\gamma,\gamma\rangle}{\langle\beta,\gamma\rangle}t\right)\right)$$

hypergeometric; also Jacobi polynomial of degree n,

$P_n^{u,u}\left(1 - 2\sin^2\left(\frac{\langle\gamma,\gamma\rangle}{\langle\beta,\gamma\rangle}t\right)\right)$ where $u = \frac{1}{2}(m_{\beta/2} + m_\beta - 1)$.

Example:

G/K noncompact riemannian symm, rank 1:

$\mathbb{F}$	X	m_γ	$m_{2\gamma}$	ρ	β
$\mathbb{R}$	$H^q(\mathbb{R})$	$q-1$	0	$\frac{1}{2}(q-1)\gamma$	γ
$\mathbb{C}$	$H^q(\mathbb{C})$	$2q-2$	1	$q\gamma$	2γ
$\mathbb{H}$	$H^q(\mathbb{H})$	$4q-4$	3	$(2q+1)\gamma$	2γ
$\mathbb{O}$	$H^2(\mathbb{O})$	8	7	11γ	2γ

The (G,K)–spherical functions:

$$\varphi_\lambda(g) = \int_K e^{(i\lambda+\rho)(A(kg))} d\mu_K(k), \lambda \in \mathfrak{a}_{\mathbb{C}}^*$$

$\varphi_\lambda = \varphi_{\lambda'}$ if and only if $\lambda' \in W(\mathfrak{g},\mathfrak{a})(\lambda)$

$$\varphi_\lambda(\exp(t\xi_0)) = {}_2F_1(\tfrac{1}{2}(a+b+1-i\lambda), \tfrac{1}{2}(a+b+1+i\lambda), a+1; -\sinh^2(z))$$

$$= \cosh^{s-u}(t)\ {}_2F_1(\tfrac{u-s-\ell+2}{2}, \tfrac{u-s}{2}, \tfrac{q\ell}{2}; \tanh^2(t))$$

Example:

$H_n = i\mathbb{R}{+}\mathbb{C}^n$ Heisenberg group: for Lie algebra

$\mathfrak{h}_n = i\mathbb{R}{+}\mathbb{C}^n$ with $[(z,u),(w,v)] = (\mathrm{Im}\,\langle u,v\rangle, 0)$

$U(n)$ is a maximal compact subgroup of $\mathrm{Aut}(H_n)$

$K = U(n)$ and $G = H_n \rtimes U(n)$ semidirect

Spherical functions:

$\varphi_0(z,v,k)) = 1,$

if $0 \neq \xi \in \mathbb{C}^n$ then

$$\varphi_\xi(z,v,k) = \int_K e^{i\mathrm{Re}\,\langle k'\xi,v\rangle} d\mu_K(k') = \frac{2^{n-1}(n-1)!}{(||\xi||\,||v||)^{n-1}} J_{n-1}(||\xi||\,||v||)$$

$$\varphi_{\zeta,m}(z,v,k) = \begin{cases} e^{i\zeta(z)} L_m^{(n-1)}(\zeta(z)||v||^2) e^{-\zeta(z)||v||^2/4} & \text{if } \zeta(i) > 0 \\ \text{or} & \\ \overline{\varphi_{-\zeta,m}(z,v,k)} & \text{if } \zeta(i) < 0, \end{cases}$$

where $L_m^{(n-1)}$ is the generalized Laguerre poly of order $n-1$, normalization $L_m^{(n-1)}(0) = 1$:

$$L_m^{(n-1)}(x) = (n-1)! \sum_{j=0}^m \begin{pmatrix} m \\ j \end{pmatrix} \frac{(-x)^j}{(j+n-1)!}$$

Induced Spherical Functions.

G: locally compact group

K: compact subgroup

Q: closed subgroup of G with $G = QK$

$\zeta : Q \to \mathbb{C}$: spherical for $(Q, Q \cap K)$

$\mathrm{Ind}_Q^G(\zeta) : G \to \mathbb{C}$ by

$\widetilde{\zeta} : G \to \mathbb{C}$ by $\widetilde{\zeta}(kq) = \zeta(q)\Delta_{G/Q}(q)^{-1/2}$

and $[\mathrm{Ind}_Q^G(\zeta)](g) = \int_K \widetilde{\zeta}(gk)d\mu_K(k)$

1. If ζ positive definite then $\mathrm{Ind}_Q^G(\zeta)$ pos def
2. If $\psi \in \widehat{Q}$ for ζ,
   then $\mathrm{Ind}_Q^G(\psi) \in \widehat{G}$ a.e. for $\mathrm{Ind}_Q^G(\zeta)$

Example: $G = KAN$ reductive, $Q = MAN$ minimal parabolic, $\alpha \in \mathfrak{a}^*$. Then $e^{i\alpha} : Q \to \mathbb{C}$ is $(Q, Q \cap K)$ spherical pos def, induced spherical fn ↔ spherical principal series rep $\mathrm{Ind}_Q^G(e^{i\alpha})$.

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Lecture 2: Harmonic Analysis

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The Spherical Transform.

The spherical transform for commutative pairs corresponds to the Fourier transform for commutative locally compact groups.

$\mathcal{S} = \mathcal{S}(G, K)$: (G, K)–spherical functions
$\mathcal{BS} = \mathcal{BS}(G, K)$: bounded spherical functions
$\mathcal{P} = \mathcal{P}(G, K)$: pos definite spherical functions

Spherical transform: map $\mathcal{F} : f \mapsto \widehat{f}$ from

$L^1(K\backslash G/K)$ to functions on $\mathcal{BS}$
$C_c(K\backslash G/K)$ to functions on $\mathcal{S}$

by $\widehat{f}(\omega) = \int_G f(g)\omega(g^{-1})\, d\mu_G(g) = m_\omega(f)$ where m_ω is the spherical measure for ω.

$\mathcal{P} \subset \mathcal{BS} \subset \mathcal{S}$: weak topol from the $\widehat{f}$, = subspace topol from $\mathcal{S} \hookrightarrow \prod_f \mathbb{C}_f$ by $\omega \mapsto (\widehat{f}(\omega))$.

$\mathcal{P} = \mathcal{P}(G, K)$ is locally compact, topology same as the subspace topology from the closed unit disk $D^* \subset L^1(K\backslash G/K)^*$, compact closure $\mathrm{cl}\,(\mathcal{P})$ in D^*, either $\mathrm{cl}\,(\mathcal{P}) = \mathcal{P}$ or $\mathrm{cl}\,(\mathcal{P}) = \mathcal{P} \cup \{0\}$.

Riemann–Lebesgue Lemma
If $f \in L^1(K\backslash G/K)$ then $\widehat{f}|_{\mathcal{P}} \in C_\infty(\mathcal{P})$.

Bochner Space: $B(K\backslash G/K)$ consists of all linear combinations of continuous positive definite K–bi–invariant functions $\phi : G \to \mathbb{C}$.

$B(K\backslash G/K) \cap L^1(K\backslash G/K)$ dense in $L^1(K\backslash G/K)$.

$C^*(G, K)$: operator norm closure of (left convolution action of) $L^1(K\backslash G/K)$ on $L^2(K\backslash G/K)$.

Tempered: $\mathcal{R} = \mathcal{R}(G, K)$ is the maximal ideal space $\mathcal{M}_{C^*(G,K)}$ of $C^*(G, K)$.

Godement–Bochner Theorem:

$\phi : G \to \mathbb{C}$ continuous pos def K–bi–invariant function there is a unique measure $\nu_\phi \in M^+(\mathcal{P})$, concentrated on $\mathcal{R} = \mathcal{R}(G,K) \subset \mathcal{P}$ such that $\phi(g) = \int_{\mathcal{P}} \omega(g)\, d\nu_\phi(\omega)$ for all $g \in G$. In particular ϕ is uniformly continuous.

$A(\mathcal{R}) := \{\widehat{f}|_{\mathcal{R}} \mid f \in L^1(K\backslash G/K)\}$ dense in $C_\infty(\mathcal{R})$

Inverse Spherical Transform: $\exists 1$ pos Radon measure $\mu_{\mathcal{P}}$ on $\mathcal{P}$, supported in $\mathcal{R}$, such that:

$\mathcal{F}(B(K\backslash G/K) \cap L^1(K\backslash G/K)) \subset L^1(\mathcal{P}, \mu_{\mathcal{P}})$,

$f(g) = \int_{\mathcal{P}} \widehat{f}(\omega)\omega(g)\, d\mu_{\mathcal{P}}(\omega)$ for $f \in B \cap L^1$.

$\mu_{\mathcal{P}}$ is **Plancherel measure** for (G,K).

Plancherel Formula for $K\backslash G/K$:

$f \in L^1(K\backslash G/K) \cap L^2(K\backslash G/K) \Rightarrow \widehat{f} \in L^2(\mathcal{P}, \mu_{\mathcal{P}})$, $||f||_2 = ||\widehat{f}||_2$, $\mathcal{F} : L^1 \cap L^2 \to L^2(\mathcal{P}, \mu_{\mathcal{P}})$ extends to isometry of $L^2(K\backslash G/K) \cong L^2(\mathcal{P}, \mu_{\mathcal{P}})$.

Direct Integral: $(Y, \mathcal{M}, \tau)$ is a measure space. If $y \in Y$ then H_y is a separable Hilbert space Fix a family $\{s_\alpha\}_{\alpha \in A}$ of maps $Y \to \bigcup_{y \in Y} H_y$ such that

(i) $s_\alpha(y) \in H_y$ a.e. (Y, τ), for all $\alpha \in A$,

(ii) $\Big(y \mapsto \langle s_\alpha(y), s_\beta(y) \rangle_{H_y}\Big) \in L^1(Y, \tau)$, and

(iii) $H_y =$ closed span $(\{s_\alpha(y)\}_{\alpha \in A})$ a.e. (Y, τ).

The corresponding **direct integral** is the vector space

$$\mathcal{H} = \int_Y H_y \, d\tau(y) :$$

$$\text{all maps } s : Y \to \bigcup_{y \in Y} H_y \text{ such that}$$

(i) $s(y) \in H_y$ a.e. (Y, τ),

(ii) $\Big(y \mapsto \langle s(y), s_\alpha(y) \rangle_{H_y}\Big) \in L^1(Y, \tau)$, all $\alpha \in A$,

inner product $\langle s, s' \rangle = \int_Y \langle s(y), s'(y) \rangle_{H_y} \, d\tau(y)$.

Every $s \in \mathcal{H} = \int_Y H_y \, d\tau(y)$ has form $s(y) = \sum_{\alpha \in A} f_\alpha(y) s_\alpha(y)$, $f_\alpha : Y \to \mathbb{C}$ measurable.

Let $\{T_y \mid y \in Y\}$ be bounded linear operators on the H_y such that if $s \in \mathcal{H} = \int_Y H_y \, d\tau(y)$ then $(T(s) : y \mapsto T_y(s(y))) \in \mathcal{H}$. Then $T : \bigcup H_y \to \mathcal{H}$ is the **direct integral** of the T_y, denoted $T = \int_Y T_y \, d\tau(y)$.

Let $\{\pi_y \in Y\}$ be a uniformly bounded family of Banach representations of G on the H_y such that each $\pi(g) = \int_Y \pi_y(g) \, d\tau(y)$ is defined. If the resulting group homomorphism $\pi : G \to GL(\mathcal{H})$ is strongly continuous, i.e. is a Banach representation of G, then π is the **direct integral** of the π_y, denoted $\pi = \int_Y \pi_y \, d\tau(y)$.

H closed subgroup of locally compact group G, η be a unitary representation of H, E_η the representation space. $\Delta_{G/H}(h) = \Delta_G(h)/\Delta_H(h)$, quotient of the modular functions. Define

$$L^2(G/H, \eta \otimes \Delta_{G/H}^{1/2}) = \\ \{\phi : G \to E_\eta \mid \ ||\phi|| \text{ is } L^2(G/H, \Delta_{G/H}^{1/2})\}.$$

So $\phi(gh) = \Delta_{G/H}(h)^{-1/2}\eta(h)^{-1}(h)\phi(g)$ and $g \mapsto ||\phi(g)||$ belongs to $L^2(G/H, \Delta_{G/H}^{1/2})$.

$L^2(G/H, \eta \otimes \Delta_{G/H}^{1/2})$ is a Hilbert space with $\langle \phi, \phi' \rangle = \int_{G/H} \langle \phi(gH), \phi'(gH) \rangle_{E_\eta} \, d\mu_{G/H}(gH)$.

G acts on $L^2(G/H, \eta \otimes \Delta_{G/H}^{1/2})$ by $[\ell_{G/H}(g)\phi](g') = \phi(g^{-1}g')$. This is a unitary representation, **unitarily induced** from H to G, denoted $\mathrm{Ind}_H^G(\eta)$.

If η is the trivial 1–dimensional representation of H then $\mathrm{Ind}_H^G(\eta \otimes \Delta_{G/H}^{1/2})$ is the left regular representation of G on $L^2(G/H, \Delta_{G/H}^{1/2})$. If ω is the trivial 1–dimensional representation of $\{1\}$ then $\mathrm{Ind}_{\{1\}}^G(\omega)$ is the usual left regular representation of G on $L^2(G)$.

$$\begin{aligned}\mathrm{Ind}_H^G\left(\left(\int_Y \eta_y \, d\tau(y)\right) \otimes \Delta_{G/H}^{1/2}\right) \\ \cong \int_Y \mathrm{Ind}_H^G\left(\eta_y \otimes \Delta_{G/H}^{1/2}\right) d\tau(y).\end{aligned}$$

Let $K, Q \subset G$ closed subgroups, K compact and transitive on G/Q, $\zeta : Q \to \mathbb{C}$ spherical for $(Q, Q \cap K)$. The **induced spherical function** $\mathrm{Ind}_Q^G(\zeta) : G \to \mathbb{C}$ by $[\mathrm{Ind}_Q^G(\zeta)](g) = \int_K \widetilde{\zeta}(gk) \, d\mu_K(k)$ where $\widetilde{\zeta}(kq) = \zeta(q)\Delta_{G/Q}(q)^{-1/2}$

Let $[\eta] \in \widehat{Q}$ with a $(Q \cap K)$–fixed cyclic unit vector v,

$\zeta(q) = \langle v, \eta(q)v\rangle$,

$\pi = \mathrm{Ind}_Q^G(\eta)$, $\phi = \mathrm{Ind}_Q^G(\zeta)$,

$u : G \longrightarrow E_\eta$ by $u(kq) = \eta(q)^{-1}\Delta_{G/Q}(q)^{-1/2}\, v$.

Then u is a K–fixed unit vector in

$H_\pi = L^2(G/Q, \eta_0 \otimes \Delta_{G/Q}^{1/2})$, and

$\phi(g) = \langle u, \pi(g)u\rangle$ for all $g \in G$.

H'_π = closed span $\{\pi(g)u \mid g \in G\}$, is π–irreducible and associated to ϕ.

We use direct integrals to carry our spherical transform results from $K\backslash G/K$ to G/K, expanding a function on G/K in terms of (i) G–translates of functions $\omega \in \mathcal{P}(G, K)$, (ii) coefficients of the corresponding representations π_ω, (iii) vectors in $H_\omega = H_{\pi_\omega}$, and (iv) operators on those representation spaces.

Bochner space $B(G/K)$: all linear combination of continuous positive–definite right K–invariant functions $\phi : G \to \mathbb{C}$.

$B(G/K) \cap L^1(G/K)$ is dense in $L^1(G/K)$.

If $f \in B(G/K) \cap L^1(G/K)$ and $g \in G$ then $\omega \mapsto (f * \omega)(g)$ is in $L^1(\mathcal{P}, \mu_{\mathcal{P}})$ and $f(g) = \int_{\mathcal{P}} (f * \omega)(g)\, d\mu_{\mathcal{P}}(\omega)$.

Scalar Fourier Inversion:
Let $f \in B(G/K) \cap L^1(G/K)$, $x \in G$. Then $(\omega \mapsto \langle \dot{\pi}_\omega(f) u_\omega, \pi_\omega(x) u_\omega \rangle_{H_\omega}) \in L^1(\mathcal{P}, \mu_{\mathcal{P}})$ and $f(x) = \int_{\mathcal{P}} \langle \dot{\pi}_\omega(f) u_\omega, \pi_\omega(x) u_\omega \rangle_{H_\omega}\, d\mu_{\mathcal{P}}(\omega)$.

Corollary: $(\omega \mapsto ||\dot{\pi}_\omega(f) u_\omega||) \in L^2(\mathcal{P}, \mu_{\mathcal{P}})$
$||f||^2_{L_2(G/K)} = \int_{\mathcal{P}} ||\dot{\pi}_\omega(f) u_\omega||^2_{H_\omega}\, d\mu_{\mathcal{P}}(\omega)$.

$\mathcal{H}^p = \mathcal{H}^p(G/K) = \int_{\mathcal{P}} H_\omega \, d\mu_{\mathcal{P}}(\omega)$:

L^p sections $s_f(\omega) = \dot{\pi}_\omega(f) u_\omega$ for $f \in C_c(G/K)$.

Vector Fourier Inversion:
Let $f \in B(G/K) \cap L^1(G/K)$. Then
$\mathcal{F}(f) \in \mathcal{H}^1(G/K)$ and
$f(x) = \int_{\mathcal{P}} \langle \mathcal{F}(f)(\omega), \pi_\omega(x) u_\omega \rangle_{H_\omega} \, d\mu_{\mathcal{P}}(\omega)$.

The **vector–valued Fourier Transform** on G/K is the map $\mathcal{F} : L^1(G/K) \to \mathcal{H}^\infty(G/K)$ defined by $[\mathcal{F}(f)](\omega) = \dot{\pi}_\omega(f) u_\omega \in H_\omega$. Note $||\mathcal{F}(f)||_{\mathcal{H}^\infty(G/K)} \leqq ||f||_{L^1(G/K)}$.

Vector Plancherel Formula:
If $f \in L^1(G/K) \cap L^2(G/K)$ then

$\mathcal{F}(f) \in \mathcal{H}^2(G/K)$,

$||\mathcal{F}(f)||_{\mathcal{H}^2(G/K)} = ||f||_{L^2(G/K)}$,

$\mathcal{F}$ extends to $L^2(G/K) \cong \mathcal{H}^2(G/K)$.

Example: Compact Groups

K compact group, δK diag in $G = K \times K$.

$K \leftrightarrow G/\delta K$ by $k \leftrightarrow (k,1)\delta K$.

G acts on K by $(k_1, k_2) : K \mapsto k_1 k k_2^{-1}$; that corresponds to action $G/\delta K$ by
$(k_1, k_2) : (k', k'')\delta K \mapsto (k_1 k', k_2 k'')\delta K$

A function $f : G \to \mathbb{C}$ is δK–bi–invariant just when $f'(xy^{-1}) = f(x,y)$ is constant on conjugacy classes in K. So $C(G/\delta K) \leftrightarrow C(K)$, $C(\delta K \backslash G/\delta K) \leftrightarrow C(K)^{\mathrm{Ad}(K)}$, etc.

If $[\tau] \in \widehat{K}$ its character $\chi_\tau(k) = \text{ trace } \tau(k)$ is conjugation–invariant, belongs to $C(\delta K \backslash G/\delta K)$.

The Inverse Spherical Transform and Plancherel Formula for $\delta K \backslash G/\delta K$ express

$L^2(\delta K \backslash G/\delta K) = \int_{\mathcal{P}} \mathbb{C}_\tau d\mu_{\mathcal{P}}(\tau)$

where $\mathbb{C}_\tau = \chi_\tau \mathbb{C}$.

Similarly the Inverse Spherical Transform and Plancherel Formula for $G/\delta K$ express

$$L^2(G/K) \cong L^2(K) \cong \textstyle\int_{\mathcal{P}} E_\tau \otimes E_\tau^* \, d\mu_{\mathcal{P}}(\tau),$$

where

E_τ is the representation space of τ,

E_τ^* its dual space,

$E_\tau \otimes E_\tau^*$ is the space of matrix coef of τ.

The considerations above for compact groups give the details:

- $\mathcal{P} = \widehat{K}$,
- Plancherel measure $\mu_{\mathcal{P}}$ is purely atomic, so $\int_{\mathcal{P}}$ is a weighted sum over $\widehat{K}$,
- $d\mu_{\mathcal{P}}(\tau) = \deg \tau$, the dimension of E_τ, and
- $f \mapsto \deg(\tau)\chi_\tau * f = f * \deg(\tau)\chi_\tau$ is the orthogonal projection $L^2(K) \to E_\tau \otimes E_\tau^*$.
- $L^2(\delta K \backslash G/\delta K) = L^2(K)^{\mathrm{Ad}(K)} = \sum_{\widehat{K}} \mathbb{C}_\tau$,
- $L^2(G/\delta K) = L^2(K) = \sum_{\widehat{K}} E_\tau \otimes E_\tau^*$,
- $f = \sum_{\widehat{K}} \deg(\tau)\chi_\tau * f$ in $L^2(K)$.

Compact Symmetric Spaces G/K:

$$\mathcal{P} = \left\{ \pi_\lambda \,\middle|\, \tfrac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}^{\geq 0} \text{ all pos res roots } \alpha \right\}$$

$$d\mu_{\mathcal{P}}(\pi_\lambda) = \deg \pi_\lambda,\ L^2(G/K) = \textstyle\sum_{\mathcal{P}} E_{\pi_\lambda}$$

$$f = \textstyle\sum_{\mathcal{P}} \deg \pi_\lambda \operatorname{trace} \dot{\pi}_\lambda(f)$$

Example: Noncompact Symmetric G/K

As in the case of rank 1, the (G,K)–spherical functions are the

$$\varphi_\lambda(g) = \textstyle\int_K e^{(i\lambda+\rho)(A(kg))} d\mu_K(k),\ \lambda \in \mathfrak{a}^*_{\mathbb{C}}$$

$\varphi_\lambda = \varphi_{\lambda'}$ if and only if $\lambda' \in W(\mathfrak{g},\mathfrak{a})(\lambda)$

Spherical inversion formula for $f \in L^1(K\backslash G/K)$:

$$cf(x) = \textstyle\int_{\mathfrak{a}^*} \widehat{f}(\lambda)\varphi_\lambda(x)|\mathbf{c}(\lambda)|^{-2} d\lambda$$

where $\mathbf{c}(\lambda) = c_0 \prod_{\gamma\in\Sigma_0^+(\mathfrak{g},\mathfrak{a})} \times$

$$\times \frac{2^{-\langle i\lambda,\gamma\rangle/\langle\gamma,\gamma\rangle}\Gamma(\frac{\langle i\lambda,\gamma\rangle}{\langle\gamma,\gamma\rangle})}{\Gamma(\frac{1}{2}(\frac{1}{2}m_\gamma+1+\frac{\langle i\lambda,\gamma\rangle}{\langle\gamma,\gamma\rangle}))\Gamma(\frac{1}{2}(\frac{1}{2}m_\gamma+m_{2\gamma}+\frac{\langle i\lambda,\gamma\rangle}{\langle\gamma,\gamma\rangle}))}$$

c_0 given by $\mathbf{c}(-i\rho) = 1$, c is independent of f,

$c \int_G |f(x)|^2 d\mu_G(x) = \int_{\mathfrak{a}^*} |\widehat{f}(\lambda)|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda$,

and $\frac{1}{c}|\mathbf{c}|^{-2}$ is the Plancherel density on $\mathfrak{a}^*$.

$L^2(G/K) = \int_{\mathfrak{a}^*} E_{\pi_\lambda} |\mathbf{c}(\lambda)|^{-2} d\lambda$

Finsler symmetric spaces:

Finsler symmetric spaces are homogeneous, and they have the same homogeneous space structure as riemannian symmetric spaces, so their harmonic analysis is exactly the same as in the riemannian case.

Short Course on Commutative Spaces
Reykjavik, August, 2007
Lecture 3: Structure and Classsification

$==========< \circ >===========$

Reductive Commutative Spaces

G is a connected reductive real Lie group and H is a compact subgroup

(G, H) is w. symm. $\Leftrightarrow$ $\exists$ **weak symmetry**:

$$\sigma \in \mathrm{Aut}(G),\ \sigma(g) \in Hg^{-1}H \ \ \forall g \in G_{\mathbb{C}}$$

(G, H), $M = G/H$ are **spherical** if a Borel subgp $B \subset G_{\mathbb{C}}$ has an open orbit on $G_{\mathbb{C}}/H_{\mathbb{C}}$

Akhiezer–Vinberg Theorem: if G is reductive then the following are equivalent.

- (G, H) is commutative
- (G, H) is weakly symmetric
- (G, H) is spherical
- G is multiplicity–free on $L^2(G/H)$
- $G_\mathbb{C}$ is multiplicity–free on $\mathbb{C}[G_\mathbb{C}]^{H_\mathbb{C}}$

The main tool in the hard part, that commutative pairs (G, H) with G reductive, is construction of a “Weyl involution” ν of G such that $\nu(H) = H$. In other words ν is an involutive automorphism of G, and G has a Cartan subgroup A such that $\nu(a) = a^{-1}$ for all $a \in A$.

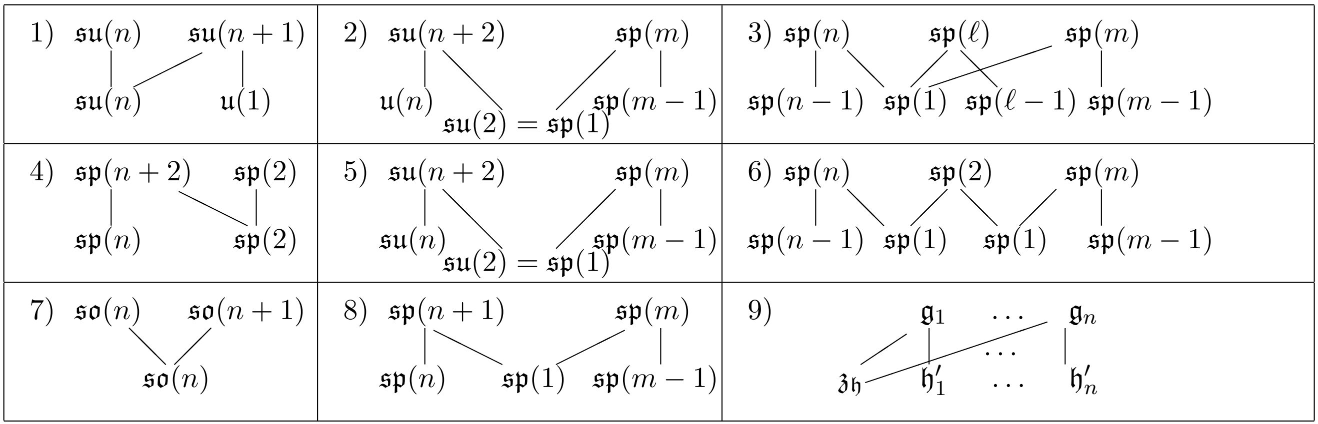
Kramer Classification.

Weakly Symmetric Coset Spaces of a Compact Connected Simple Lie Group				
$M = G/H$ weakly symmetric				G/K symmetric
	G	H	conditions	K with $H \subset K \subset G$
riemannian symmetric spaces with symmetry s				$(H = K)$
circle bundles over hermitian symmetric spaces dual to a non–tube domain:				
1	$SU(m+n)$	$SU(m) \times SU(n)]$	$n > m \geqq 1$	$S[U(m) \times U(n)]$
2	$SO(2n)$	$SU(n)$	n odd, $n \geqq 3$	$U(n)$
3	E_6	$Spin(10)$		$Spin(10) \cdot Spin(2)$
4	$SU(2n+1)$	$Sp(n)$	$n \geqq 1$	$U(2n) = S[U(2n) \times U(1)]$
5	$SU(2n+1)$	$Sp(n) \times U(1)$	$n \geqq 1$	$U(2n) = S[U(2n) \times U(1)]$
constant positive curvature spheres:				
6	$Spin(7)$	G_2		(there is none)
7	G_2	$SU(3)$		(there is none)
weakly symmetric spaces of Cayley type:				
8	$SO(10)$	$Spin(7) \times SO(2)$		$SO(8) \times SO(2)$
9	$SO(9)$	$Spin(7)$		$SO(8)$
10	$Spin(8)$	G_2		$Spin(7)$
11	$SO(2n+1)$	$U(n)$	$n \geqq 2$	$SO(2n)$
12	$Sp(n)$	$Sp(n-1) \times U(1)$	$n \geqq 1$	$Sp(n-1) \times Sp(1)$

Weakly Symmetric Coset Spaces of a Noncompact Connected Simple Lie Group				
$M = G/H$ weakly symmetric				max compact K in G
	G	H	conditions	K
riemannian symmetric spaces				$(K = H)$
circle bundles over bounded symmetric domains:				
1	$SU(m,n)$	$SU(m) \times SU(n)]$	$n > m \geqq 1$	$S[U(m) \times U(n)]$
2	$SO^*(2n)$	$SU(n)$	n odd, $n \geqq 3$	$U(n)$
3	E_{6,D_5T_1}	$Spin(10)$		$Spin(10) \cdot Spin(2)$
4	$SU(2n,1)$	$Sp(n)$	$n \geqq 1$	$U(2n) = S[U(2n) \times U(1)]$
5	$SU(2n,1)$	$Sp(n) \times U(1)$	$n \geqq 1$	$U(2n) = S[U(2n) \times U(1)]$
weakly symmetric spaces of Cayley type:				
8	$SO(8,2)^0$	$Spin(7) \times SO(2)$		$SO(8) \times SO(2)$
9	$SO(8,1)^0$	$Spin(7)$		$SO(8)$
10	$Spin(7,1)^0$	G_2		$Spin(7)$
11	$SO(2n,1)^0$	$U(n)$	$n \geqq 2$	$SO(2n)$
12	$Sp(n-1,1)$	$Sp(n-1) \times U(1)$	$n \geqq 1$	$Sp(n-1) \times Sp(1)$

Yakimova–Mikityuk Classification.

Compact Irreducible Nonsymmetric Weakly Symmetric $(\mathfrak{g},\mathfrak{h})$, where $\mathfrak{g}$ is Semisimple but not Simple



($\mathfrak{g}$ is the sum of the algebras on the top row and $\mathfrak{h}$ is the sum of the algebras on the bottom row)

Noncompact version: Make $\geqq 1$ of these for simple ideals of $\mathfrak{g}$, so that the resulting real form $\mathfrak{g}_0$ of $\mathfrak{g}_\mathbb{C}$ still contains $\mathfrak{h}$.

1. If $\mathfrak{m}$ is a simple ideal of $\mathfrak{h}$ and $\mathfrak{g}$ has an ideal $\mathfrak{l}$ isomorphic to $\mathfrak{m}\oplus\mathfrak{m}$, such that $\mathfrak{h}\hookrightarrow\mathfrak{g}$ restricts to the inclusion of $\mathfrak{m}$ as the diagonal of $\mathfrak{l}\cong\mathfrak{m}\oplus\mathfrak{m}$, replace $\mathfrak{l}$ by $\mathfrak{m}_\mathbb{C}$.
2. Replace a simple summand of $\mathfrak{g}$ not involved in the first kind of substitution with a noncompact real form of its complexification, and there is only one possibility for that noncompact real form.

Commutative Nilmanifolds

The 2–step Nilpotent Theorem. If (G, K) is commutative and N is the nilradical of G, then N is abelian or 2–step nilpotent.

Lemma: $(N \rtimes K, K)$ is commutative.

Lie group argument: Use $KnKn'K = Kn'KnK$ and Campbell–Hausdorff formula on the descending central series of N (Benson-Jenkins-Ratcliff)

Geometric argument (weakly symmetric case):
Weakly symmetric ⇒ geodesic orbit space
(Berndt, Kowalsky, Vanhecke)
(N, ds^2) g.o. riemannian nilmanifold ⇒ N abelian
or 2–step nilpotent (Gordon)

Algebraic + symplectic geometry argument: $G = N \rtimes L$ where (i) N abelian or 2–step nilpotent, (ii) L reductive and $K \subset L$, and (iii) $\mathbb{R}[\mathfrak{n}]^L = \mathbb{R}[\mathfrak{n}]^K$ (Vinberg)

Carcano's Theorem. Let $K \subset U(n)$ closed connected subgroup irreducible on $\mathbb{C}^n$. Then $(H_n \rtimes K, K)$ commutative $\Leftrightarrow$ $K_{\mathbb{C}}$ multiplicity free on polynomial algebra $\mathbb{C}[\mathbb{C}^n]$.

Kač' Theorem. Classification of closed connected irreducible subgroups $K \subset U(n)$ multiplicity free on $\mathbb{C}[\mathbb{C}^n]$.

Benson–Jenkins–Ratcliff Classification. Let $K \subset U(n)$ closed connected subgroup irreducible on $\mathbb{C}^n$. Then $(H_n \rtimes K, K)$ commutative $\Leftrightarrow$ $K \subset U(n)$ multiplicity free on $\mathbb{C}[\mathbb{C}^n]$ $\Leftrightarrow$ representation of K on $\mathbb{C}^n$ is one of the following.

"Multiplicity Free" Irreducible Representations of K and $K_{\mathbb{C}}$ on $\mathbb{C}^n$

	Group K	Group $K_{\mathbb{C}}$	Acting on	Conditions on n
1	$SU(n)$	$SL(n;\mathbb{C})$	$\mathbb{C}^n$	$n \geqq 2$
2	$U(n)$	$GL(n;\mathbb{C})$	$\mathbb{C}^n$	$n \geqq 1$
3	$Sp(m)$	$Sp(m;\mathbb{C})$	$\mathbb{C}^n$	$n = 2m$
4	$U(1) \times Sp(m)$	$\mathbb{C}^* \times Sp(m;\mathbb{C})$	$\mathbb{C}^n$	$n = 2m$
5	$U(1) \times SO(n)$	$\mathbb{C}^* \times SO(n;\mathbb{C})$	$\mathbb{C}^n$	$n \geqq 2$
6	$U(m)$	$GL(m;\mathbb{C})$	$S^2(\mathbb{C}^m)$	$m \geqq 2,\ n = \frac{1}{2}m(m+1)$
7	$SU(m)$	$SL(m;\mathbb{C})$	$\Lambda^2(\mathbb{C}^m)$	m odd, $n = \frac{1}{2}m(m-1)$
8	$U(m)$	$GL(m;\mathbb{C})$	$\Lambda^2(\mathbb{C}^m)$	$n = \frac{1}{2}m(m-1)$
9	$SU(\ell) \times SU(m)$	$SL(\ell;\mathbb{C}) \times SL(m;\mathbb{C})$	$\mathbb{C}^\ell \otimes \mathbb{C}^m$	$n = \ell m,\ \ell \neq m$
10	$U(\ell) \times SU(m)$	$GL(\ell;\mathbb{C}) \times SL(m;\mathbb{C})$	$\mathbb{C}^\ell \otimes \mathbb{C}^m$	$n = \ell m$
11	$U(2) \times Sp(m)$	$GL(2;\mathbb{C}) \times Sp(m;\mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^{2m}$	$n = 4m$
12	$SU(3) \times Sp(m)$	$SL(3;\mathbb{C}) \times Sp(m;\mathbb{C})$	$\mathbb{C}^3 \otimes \mathbb{C}^{2m}$	$n = 6m$
13	$U(3) \times Sp(m)$	$GL(3;\mathbb{C}) \times Sp(m;\mathbb{C})$	$\mathbb{C}^3 \otimes \mathbb{C}^{2m}$	$n = 6m$
14	$U(4) \times Sp(4)$	$GL(4;\mathbb{C}) \times Sp(4;\mathbb{C})$	$\mathbb{C}^4 \otimes \mathbb{C}^8$	$n = 32$
15	$SU(m) \times Sp(4)$	$SL(m;\mathbb{C}) \times Sp(4;\mathbb{C})$	$\mathbb{C}^m \otimes \mathbb{C}^8$	$n = 8m,\ m \geqq 3$
16	$U(m) \times Sp(4)$	$GL(m;\mathbb{C}) \times Sp(4;\mathbb{C})$	$\mathbb{C}^m \otimes \mathbb{C}^8$	$n = 8m,\ m \geqq 3$
17	$U(1) \times Spin(7)$	$\mathbb{C}^* \times Spin(7;\mathbb{C})$	$\mathbb{C}^8$	$n = 8$
18	$U(1) \times Spin(9)$	$\mathbb{C}^* \times Spin(9;\mathbb{C})$	$\mathbb{C}^{16}$	$n = 16$
19	$Spin(10)$	$Spin(10;\mathbb{C})$	$\mathbb{C}^{16}$	$n = 16$
20	$U(1) \times Spin(10)$	$\mathbb{C}^* \times Spin(10;\mathbb{C})$	$\mathbb{C}^{16}$	$n = 16$
21	$U(1) \times G_2$	$\mathbb{C}^* \times G_{2,\mathbb{C}}$	$\mathbb{C}^7$	$n = 7$
22	$U(1) \times E_6$	$\mathbb{C}^* \times E_{6,\mathbb{C}}$	$\mathbb{C}^{27}$	$n = 27$

If (G, K) is one of the pairs in this table, and ds^2 is any G–invariant riemannian metric on $M = G/K$, then (G, K) and (M, ds^2) are weakly symmetric.

Vinberg Classification.[1]

Maximal Irreducible Nilpotent Gelfand Pairs $(N \rtimes K, K)$					
	Group K	$\mathfrak{v}$	$\mathfrak{z}$	$U(1)$	maximal
1	$SO(n)$	$\mathbb{R}^n$	$\mathrm{Skew}\,\mathbb{R}^{n\times n} = \mathfrak{so}(n)$		
2	$Spin(7)$	$\mathbb{R}^8 = \mathbb{O}$	$\mathbb{R}^7 = \mathrm{Im}\,\mathbb{O}$		
3	G_2	$\mathbb{R}^7 = \mathrm{Im}\,\mathbb{O}$	$\mathbb{R}^7 = \mathrm{Im}\,\mathbb{O}$		
4	$U(1) \cdot SO(n)$	$\mathbb{C}^n$	$\mathrm{Im}\,\mathbb{C}$		$n \neq 4$
5	$(U(1)\times)SU(n)$	$\mathbb{C}^n$	$\Lambda^2\mathbb{C}^n \oplus \mathrm{Im}\,\mathbb{C}$	n odd	
6	$SU(n), n$ odd	$\mathbb{C}^n$	$\Lambda^2\mathbb{C}^n$		
7	$SU(n), n$ odd	$\mathbb{C}^n$	$\mathrm{Im}\,\mathbb{C}$		
8	$U(n)$	$\mathbb{C}^n$	$\mathrm{Im}\,\mathbb{C}^{n\times n} = \mathfrak{u}(n)$		
9	$(U(1)\cdot)Sp(n)$	$\mathbb{H}^n$	$\mathrm{Re}\,\mathbb{H}_0^{n\times n} \oplus \mathrm{Im}\,\mathbb{H}$		
10	$U(n)$	$S^2\mathbb{C}^n$	$\mathbb{R}$		
11	$(U(1)\cdot)SU(n), n \geqq 3$	$\Lambda^2\mathbb{C}^n$	$\mathbb{R}$	n even	
12	$U(1) \cdot Spin(7)$	$\mathbb{C}^8$	$\mathbb{R}^7 \oplus \mathbb{R}$		
13	$U(1) \cdot Spin(9)$	$\mathbb{C}^{16}$	$\mathbb{R}$		
14	$(U(1)\cdot)Spin(10)$	$\mathbb{C}^{16}$	$\mathbb{R}$		
15	$U(1) \cdot G_2$	$\mathbb{C}^7$	$\mathbb{R}$		
16	$U(1) \cdot E_6$	$\mathbb{C}^{27}$	$\mathbb{R}$		
17	$Sp(1) \times Sp(n)$	$\mathbb{H}^n$	$\mathrm{Im}\,\mathbb{H} = \mathfrak{sp}(1)$		$n \geqq 2$
18	$Sp(2) \times Sp(n)$	$\mathbb{H}^{2\times n}$	$\mathrm{Im}\,\mathbb{H}^{2\times 2} = \mathfrak{sp}(2)$		
19	$(U(1)\cdot)SU(m) \times SU(n)$ $m, n \geqq 3$	$\mathbb{C}^m \otimes \mathbb{C}^n$	$\mathbb{R}$	$m = n$	
20	$(U(1)\cdot)SU(2) \times SU(n)$	$\mathbb{C}^2 \otimes \mathbb{C}^n$	$\mathrm{Im}\,\mathbb{C}^{2\times 2} = \mathfrak{u}(2)$	$n = 2$	
21	$(U(1)\cdot)Sp(2) \times SU(n)$	$\mathbb{H}^2 \otimes \mathbb{C}^n$	$\mathbb{R}$	$n \leqq 4$	$n \geqq 3$
22	$U(2) \times Sp(n)$	$\mathbb{C}^2 \otimes \mathbb{H}^n$	$\mathrm{Im}\,\mathbb{C}^{2\times 2} = \mathfrak{u}(2)$		
23	$U(3) \times Sp(n)$	$\mathbb{C}^3 \otimes \mathbb{H}^n$	$\mathbb{R}$		$n \geqq 2$

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[1] $\mathrm{Im}\,\mathbb{F}^{n\times n} := \{x \in \mathbb{F}^{n\times n} \mid x^* + x = 0\}$, $\mathrm{Re}\,\mathbb{F}^{n\times n} := \{x \in \mathbb{F}^{n\times n} \mid x^* = x\}$, $\mathrm{Re}\,\mathbb{F}_0^{n\times n} := \{x \in \mathrm{Re}\,\mathbb{F}^{n\times n} \mid \ \mathrm{trace}\, x = 0\}$. All are weakly symmetric except (9) with $K = Sp(n)$ (Lauret, first example of non-ws commutative pair).

Yakimova Classification.

Maximal Indecomposable Principal Saturated Nilpotent Gelfand Pairs $(N \rtimes K, K)$,
N Nonabelian, Where the Action of K on $\mathfrak{v} \cong \mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$ is Reducible

	Group K	K–module $\mathfrak{v}$	K–module $[\mathfrak{n},\mathfrak{n}]$	Algebra $\mathfrak{n}$
1	$U(n)$	$\mathbb{C}^n \oplus \mathfrak{su}(n)$	$\mathbb{R}$	$((\mathfrak{h}_{n;\mathbb{C}})) + \mathfrak{su}(n)$
2	$U(4)$	$\mathbb{C}^4 \oplus \mathbb{R}^6$	$\operatorname{Im}\mathbb{C} \oplus \Lambda^2\mathbb{C}^4$	$((\operatorname{Im}\mathbb{C} + \Lambda^2\mathbb{C}^4 + \mathbb{C}^4)) + \mathbb{R}^6$
3	$U(1) \times U(n)$	$\mathbb{C}^n \oplus \Lambda^2\mathbb{C}^n$	$\mathbb{R} \oplus \mathbb{R}$	$((\mathfrak{h}_{n;\mathbb{C}})) + ((\mathfrak{h}_{n(n-1)/2;\mathbb{C}}))$
4	$SU(4)$	$\mathbb{C}^4 \oplus \mathbb{R}^6$	$\operatorname{Im}\mathbb{C} \oplus \Lambda^2\mathbb{C}^4$	$((\operatorname{Im}\mathbb{C} + \Lambda^2\mathbb{C}^4 + \mathbb{C}^4)) + \mathbb{R}^6$
5	$U(2) \times U(4)$	$\mathbb{C}^{2\times 4} \oplus \mathbb{R}^6$	$\operatorname{Im}\mathbb{C}^{2\times 2}$	$((\operatorname{Im}\mathbb{C}^{2\times 2} + \mathbb{C}^{2\times 4})) + \mathbb{R}^6$
6	$S(U(4) \times U(m))$	$\mathbb{C}^{4\times m} \oplus \mathbb{R}^6$	$\mathbb{R}$	$((\mathfrak{h}_{4m;\mathbb{C}})) + \mathbb{R}^6$
7	$U(m) \times U(n)$	$\mathbb{C}^{m\times n} \oplus \mathbb{C}^m$	$\mathbb{R} \oplus \mathbb{R}$	$((\mathfrak{h}_{mn;\mathbb{C}})) + ((\mathfrak{h}_{m;\mathbb{C}}))$
8	$U(1) \times Sp(n) \times U(1)$	$\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$	$\mathbb{R} \oplus \mathbb{R}$	$((\mathfrak{h}_{2n;\mathbb{C}})) + ((\mathfrak{h}_{2n;\mathbb{C}}))$
9	$Sp(1) \times Sp(n) \times U(1)$	$\mathbb{H}^n \oplus \mathbb{H}^n$	$\operatorname{Im}\mathbb{H} \oplus \mathbb{R}$	$((\mathfrak{h}_{n;\mathbb{H}})) + ((\mathfrak{h}_{2n;\mathbb{C}}))$
10	$Sp(1) \times Sp(n) \times Sp(1)$	$\mathbb{H}^n \oplus \mathbb{H}^n$	$\operatorname{Im}\mathbb{H} \oplus \operatorname{Im}\mathbb{H}$	$((\mathfrak{h}_{n;\mathbb{H}})) + ((\mathfrak{h}_{n;\mathbb{H}}))$
11	$Sp(n) \times \{Sp(1), U(1), \{1\}\} \times Sp(m)$	$\mathbb{H}^n \oplus \mathbb{H}^{n\times m}$	$\operatorname{Im}\mathbb{H}$	$((\mathfrak{h}_{n;\mathbb{H}})) + \mathbb{H}^{n\times m}$
12	$Sp(n) \times \{Sp(1), U(1), \{1\}\}$	$\mathbb{H}^n \oplus \operatorname{Re}\mathbb{H}_0^{n\times n}$	$\operatorname{Im}\mathbb{H}$	$((\mathfrak{h}_{n;\mathbb{H}})) + \operatorname{Re}\mathbb{H}_0^{n\times n}$
13	$Spin(7) \times \{SO(2), \{1\}\}$	$(\mathbb{R}^8 = \mathbb{O}) \oplus \mathbb{R}^{7\times 2}$	$\mathbb{R}^7 = \operatorname{Im}\mathbb{O}$	$((\mathfrak{h}_{1;\mathbb{O}})) + \mathbb{R}^{7\times 2}$
14	$U(1) \times Spin(7)$	$\mathbb{C}^7 \oplus \mathbb{R}^8$	$\mathbb{R}$	$((\mathfrak{h}_{7;\mathbb{C}})) + \mathbb{R}^8$
15	$U(1) \times Spin(7)$	$\mathbb{C}^8 \oplus \mathbb{R}^7$	$\mathbb{R}$	$((\mathfrak{h}_{8;\mathbb{C}})) + \mathbb{R}^7$
16	$U(1) \times U(1) \times Spin(8)$	$\mathbb{C}^8_+ \oplus \mathbb{C}^8_-$	$\mathbb{R} \oplus \mathbb{R}$	$((\mathfrak{h}_{8;\mathbb{C}})) + ((\mathfrak{h}_{8;\mathbb{C}}))$
17	$U(1) \times Spin(10)$	$\mathbb{C}^{16} \oplus \mathbb{R}^{10}$	$\mathbb{R}$	$((\mathfrak{h}_{16;\mathbb{C}})) + \mathbb{R}^{10}$
18	$\{SU(n), U(n), U(1)Sp(\frac{n}{2})\} \times SU(2)$	$\mathbb{C}^{n\times 2} \oplus \mathfrak{su}(2)$	$\mathbb{R}$	$((\mathfrak{h}_{2n;\mathbb{C}})) + \mathfrak{su}(2)$
19	$\{SU(n), U(n), U(1)Sp(\frac{n}{2})\} \times U(2)$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^2$	$\mathbb{R} \oplus \mathbb{R}$	$((\mathfrak{h}_{2n;\mathbb{C}})) + ((\mathfrak{h}_{2;\mathbb{C}}))$
20	$\{SU(n), U(n), U(1)Sp(\frac{n}{2})\} \times SU(2) \times \{SU(m), U(m), U(1)Sp(\frac{m}{2})\}$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^{2\times m}$	$\mathbb{R} \oplus \mathbb{R}$	$((\mathfrak{h}_{2n;\mathbb{C}})) + ((\mathfrak{h}_{2m;\mathbb{C}}))$
21	$\{SU(n), U(n), U(1)Sp(\frac{n}{2})\} \times SU(2) \times U(4)$	$\mathbb{C}^{n\times 2} \oplus \mathbb{C}^{2\times 4} \oplus \mathbb{R}^6$	$\mathbb{R} \oplus \mathbb{R}$	$((\mathfrak{h}_{2n;\mathbb{C}})) + ((\mathfrak{h}_{8;\mathbb{C}})) + \mathbb{R}^6$
22	$U(4) \times U(2)$	$\mathbb{R}^6 \oplus \mathbb{C}^{4\times 2} \oplus \mathfrak{su}(2)$	$\mathbb{R}$	$\mathbb{R}^6 + ((\mathfrak{h}_{8;\mathbb{C}})) + \mathfrak{su}(2)$
23	$U(4) \times U(2) \times U(4)$	$\mathbb{R}^6 \oplus \mathbb{C}^{4\times 2} \oplus \mathbb{C}^{2\times 4} \oplus \mathbb{R}^6$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}^6 + ((\mathfrak{h}_{8;\mathbb{C}})) + ((\mathfrak{h}_{8;\mathbb{C}})) + \mathbb{R}^6$
24	$U(1) \times U(1) \times SU(4)$	$\mathbb{C}^4 \oplus \mathbb{C}^4 \oplus \mathbb{R}^6$	$\mathbb{R} \oplus \mathbb{R}$	$((\mathfrak{h}_{4;\mathbb{C}})) + ((\mathfrak{h}_{4;\mathbb{C}})) + \mathbb{R}^6$
25	$(U(1)\cdot)SU(4)(\cdot SO(2))$	$\mathbb{C}^4 \oplus \mathbb{R}^{6\times 2}$	$\mathbb{R}$	$((\mathfrak{h}_{4;\mathbb{C}})) + \mathbb{R}^{6\times 2}$

Square Integrable Representations.

N: connected simply connected nilpotent

Z: center of N

Equivalent:
1. N has L^2 (mod Z) unitary representations
2. $\exists\lambda \in \mathfrak{z}^*$ with $Pf(b_\lambda) \neq 0$
3. Plancherel measure on $\widehat{N}$ concentrated on $\{\pi_\lambda \mid Pf(b_\lambda) \neq 0\}$; $|Pf(b_\lambda)|$ is the Plancherel density there.

Example: $N = H_n$ Heisenberg of dim $2n+1$. Then $\mathfrak{h}_n = \mathfrak{n} = \mathfrak{z} + \mathfrak{v}$, $\mathfrak{z} = \operatorname{Im}\mathbb{C}$, $\mathfrak{v} = \mathbb{C}^n$. Let $\lambda_t : (z, v) \mapsto -itz$. Then $Pf(b_{\lambda_t}) = |t|^n$.

All the groups N of Vinberg's table have square integrable representations, except for (1) and (6) with n odd, and (3). All the groups N of Yakimova's table have square integrable representations.

General Classification

Yakimova's classification of commutative Lie group pairs is explicit under two technical conditions: "saturated" and "principal". For what it's worth without the definitions, her result is

Every maximal, indecomposable, principal, $Sp(1)$–saturated commutative pair is one of
(1) Gelfand pairs (G, K) with G reductive;
(2) one of the spaces

L	K	$\mathfrak{n}$	L	K	$\mathfrak{n}$
$(S)U(2n)$	$Sp(n)(\cdot U(1))$	$\mathfrak{h}_{2n}$ or $\mathbb{C}^{2n}$	$SO(8) \times SO(2)$	$Spin(7) \times SO(2)$	$\mathfrak{h}_8$ or $\mathbb{R}^{8\times 2}$
$SO(7)$	G_2	$\mathbb{R}^7$	$SO(8)$	$Spin(7)$	$\mathbb{R}^{8\times 2}$
$Spin(7)$	$Spin(6)$	$\mathbb{R}^8$	$SO(8)$	$Spin(7)$	$\mathbb{R}^8$
$SO(2n)$	$U(n)$	$\mathbb{R}^{2n}$			

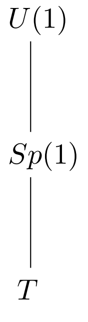
(3) one of the spaces

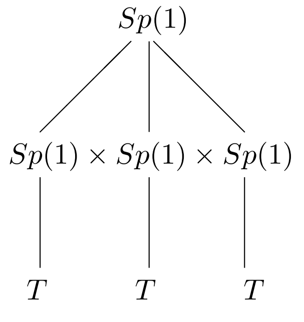
	G	K	L
(3a)	$\mathbb{R}^n \rtimes (SO(n) \times SO(n)$	$SO(n)$	$SO(n) \times SO(n)$
(3b)	$\mathbb{H}^{n\times 2} \rtimes (Sp(n) \times Sp(2) \times Sp(2))$	$Sp(n) \times Sp(2)$	$Sp(n) \times Sp(2) \times Sp(2)$
(3c)	$(H_n \rtimes S(U(n)) \times U(n))$	$U(n)$	$S(U(n) \times U(n))$

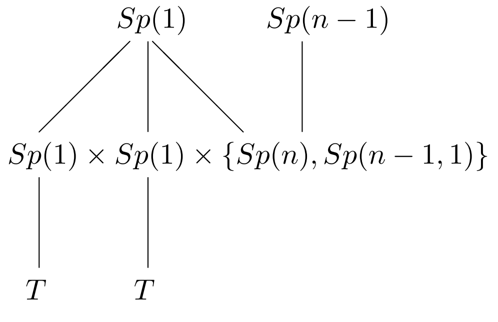
(4) pairs $(G, K) = (N \rtimes K, K)$, N nilpotent

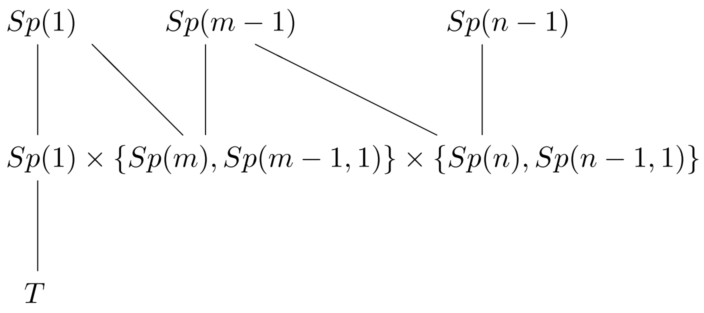
To get rid of the $Sp(1)$–saturated condition one concretely associates families of commutative spaces to certain weighted trees T, and some closely related weighted graphs Γ. A specific procedure is developed for this. The result is:

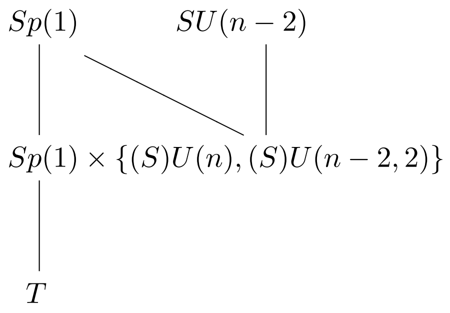
(G, K): maximal principal indecomposable Gelfand pair not $Sp(1)$–saturated, G is not reductive, $G \neq N \rtimes K$ with N nilpotent. Then (G, K) belongs to one of these eleven classes. Top row is K, the second row is L, the bottom row refers to the weighted graph (Γ, w) and/or the weighted rooted tree (T, w).

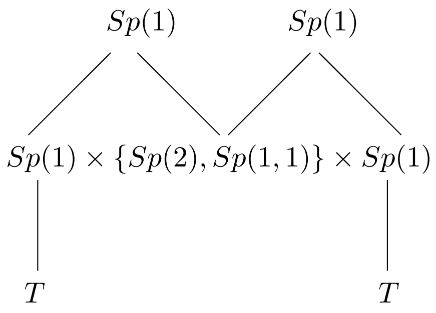


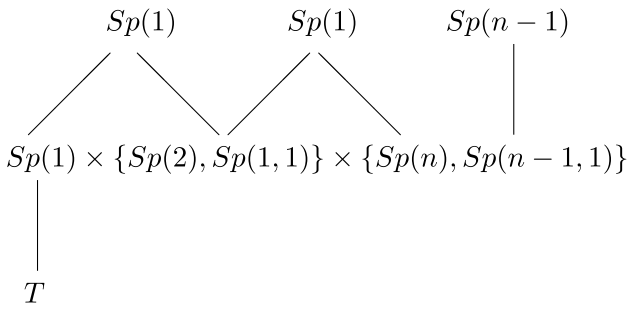


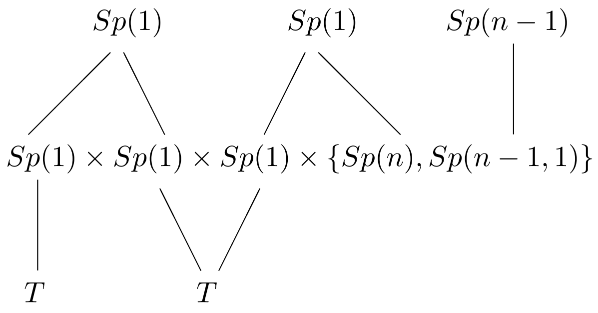


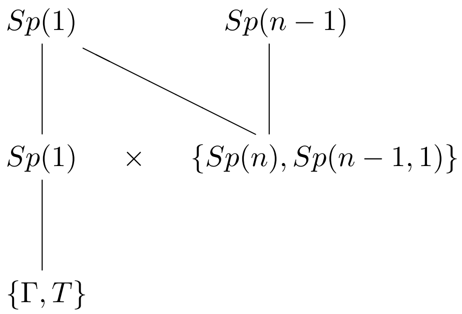


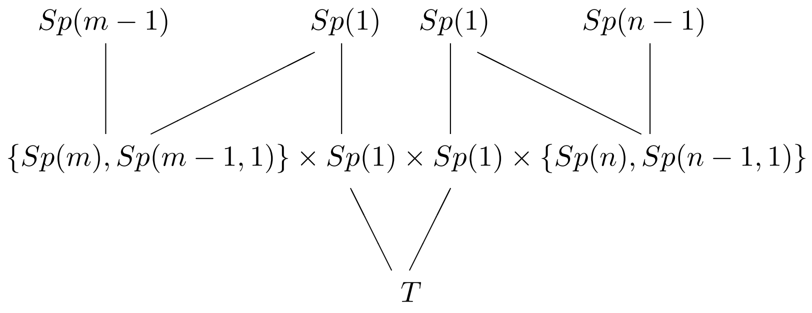


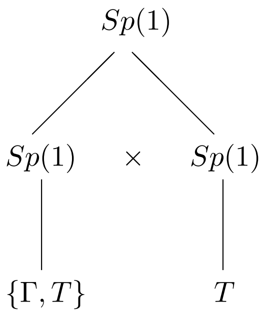












Similar considerations for the reductive and the nilmanifold cases, and again similar considerations on how to enlarge the center of L to eliminate the "principal" condition.