Polar actions and characterizations of symmetric spaces

Gudlaugur Thorbergsson

Mathematisches Institut
Universität zu Köln

International Conference on
Integral Geometry, Harmonic Analysis and
Representation Theory
in honor of
Sigurdur Helgason
on the occasion of his 80th birthday

Reykjavík
August, 15 - 18, 2007

Polar Actions

Definition (Polar Actions)

One says that an isometric action of a Lie group G on a Riemannian manifold V is **polar** if there is a complete immersed submanifold Σ in V

1. which meets all orbits of G
2. such that all intersections between Σ and orbits are perpendicular.

The submanifold Σ is called a **section** of the action.

The action is called **hyperpolar** if the section is flat.

Remark

One should thing of a section as a **set of canonical forms** for the action.
It is rather easy to see that a section is totally geodesic.

Examples

(1) Any isometric action with a hypersurface as an orbit is polar since a geodesic which meets one orbit orthogonally meets all orbits orthogonally and is hence a section.

(2) Let V be the linear space $\mathcal{S}(n)$ consisting of real symmetric $n \times n$-matrices endowed with the scalar product

$$\langle X, Y \rangle = \mathrm{trace}(XY).$$

Let G be the group $\mathrm{SO}(n)$ acting on V by conjugation. We let Σ denote the diagonal matrices in V.

Then we know from linear algebra that every matrix X in V can be conjugated into Σ by an element of G.

It is now easy to show that the intersections of conjugacy classes of matrices in V with Σ are all perpendicular.

The action is therefore hyperpolar.

Examples

(3) Let V be a compact connected Lie group G with a bi-invariant Riemannian metric acting on itself by conjugation.

Let Σ be a maximal torus in G.

The theorem on maximal tori says that all conjugacy classes in G meet Σ.

It is easy to show that the intersections between conjugacy classes in G and Σ are all perpendicular.

It follows that the action is hyperpolar since Σ is flat.

Examples

(4) Let V be a symmetric space
Let Σ be a maximal flat and totally geodesic submanifold passing through p_0 in V.
Let K denote the isometries of V that fix p_0.
Then the action of K on V is hyperpolar with Σ as a section.

This example generalizes (3) since a compact connected Lie group K with a bi-invariant Riemannian metric is a symmetric space with a maximal torus as a maximal flat and totally geodesic submanifold.

Examples

(cont.) Now the action of K on V induces an action of K on the tangent space $T_{p_0}V$ which is called the *isotropy representation of the symmetric space V.* This isotropy representation is polar with $T_{p_0}\Sigma$ as a section.

The example in (2) is a special case and corresponds to the symmetric space

$$V = \mathrm{GL}(n, \mathbf{R})/\mathrm{SO}(n).$$

One clearly has the following direct sum decomposition

$$\mathfrak{gl}(n, \mathbf{R}) = \mathfrak{so}(n) \oplus \mathcal{S}(n)$$

into skew and symmetric matrices, and this decomposition is invariant under $\mathrm{Ad}_G(K)$. Hence one can identify $T_{p_0}V$ with $\mathcal{S}_0(n)$.

Examples

(5) One can generalize the action of K on the symmetric space $V = G/K$ as follows.
Assume that (G, K_1) and (G, K_2) are symmetric pairs. Then one can show that the action of K_1 on $V = G/K_2$ is hyperpolar.

This example was introduced by Hermann and we will refer to it as a **Hermann action.**

One gets concrete examples of this kind by considering Grassmann manifolds

$$G_k(\mathbf{C}^n) = \mathrm{SU}(n+1)/K_k$$

where K_k is the stabilizer of $\mathbf{C}^k$ in $\mathbf{C}^n$. The actions of the groups $K_1, \ldots, K_{n-1}$ on $G_k(\mathbf{C}^n)$ are all hyperpolar.

Examples

(6) An example of a *polar action which is not hyperpolar:*

We let V be the complex projective space $P^n(\mathbf{C})$ endowed with the Fubini-Study metric which is invariant under the action of $\mathrm{SU}(n+1)$.

Let T^n be the maximal torus in $\mathrm{SU}(n+1)$ consisting of diagonal matrices.

It is not difficult to see that the action of T^n on $P^n(\mathbf{C})$ is polar with $P^n(\mathbf{R})$ as a section.

This action is of course not hyperpolar since any two sections of a polar actions are isometric and there can therefore not be a flat section.

Let K_1 and K_2 act isometrically on V_1 and V_2 respectively.

Then the actions of K_1 and K_2 are said to be **orbit equivalent** if there is an isometry $f : V_1 \to V_2$ that maps the orbits of K_1 onto the orbits of K_2.

Theorem (Dadok)

Let K be a compact group acting in a polar fashion on a Euclidean space V. Then the action of K is orbit equivalent to the isotropy representation of some symmetric space.

The **cohomogeneity** of an action is the minimal codimension of its orbits.

Theorem (Kollross)

Let $V = G/K$ be a compact **irreducible** *symmetric space and let H be a subgroup of G which acts in a hyperpolar fashion on V with cohomogeneity at least two. Then the action of H on V is orbit equivalent to a Hermann action.*

Kollross also classified cohomogeneity one actions on compact irreducible symmetric spaces V. Such actions on spheres were already classified by Hsiang and Lawson.

Theorem (Kollross)

Let V be a compact symmetric space of rank greater than one whose isometry group G is **simple**. *Let H be a closed connected nontrivial subgroup of G whose action on V is polar. Then the action of H is hyperpolar.*

Conjecture

Let V be a compact Riemannian manifold with **positive sectional curvature** *on which a compact Lie group G acts in a polar fashion with cohomogeneity* **at least two.**
Then V is homeomorphic to a compact rank one symmetric space M and the homeomorphism conjugates the action on V to a polar action on M.

Conjecture

Let V be an irreducible compact Riemannian manifold with **nonnegative sectional curvature** *on which a compact Lie group G acts in a hyperpolar fashion with cohomogeneity* **at least two.**
Then V is a symmetric space.

Remark

There are examples of polar actions on manifolds which are not homeomorphic to symmetric spaces. These examples are obtained by gluing constructions and blow ups.

Remark

Polar actions on compact rank one symmetric spaces are classified: On spheres by Dadok and on projective spaces by Podestà and myself.

Definition

Let Σ be a section of a polar action of a Lie group G on a Riemannian manifold V.
Then the **(generalized) Weyl group** W of the action is defined to be $W = N_G(\Sigma)/Z_G(\Sigma)$ where $N_G(\Sigma)$ is the normalizer and $Z_G(\Sigma)$ the centralizer of Σ in G.

On can show that $M/G = \Sigma/W$.

Remark

If G is a compact Lie group acting on itself by conjugations, then the generalized Weyl group is the Weyl group of the Lie group.
Similarly if K acts on the symmetric space G/K, then the generalized Weyl group is the Weyl group of the symmetric space.

Singular Riemannian Foliations

Definition

A partition $\mathcal{F}$ of a Riemannian manifold V into differentiable submanifolds, called **leaves**, is called a **singular Riemannian foliation** if

1. $\mathcal{F}$ is a singular foliation, i.e.,

$$T_pL = \{X_p \,|\, X \in \mathcal{X}_{\mathcal{F}}(V)\},$$

   where $\mathcal{X}_{\mathcal{F}}(V)$ denotes the space of vector fields on V with values tangential to the submanifolds in $\mathcal{F}$,
2. and $\mathcal{F}$ is transnormal, i.e., every geodesic that meets a leaf in $\mathcal{F}$ orthogonally meets all leaves orthogonally.

Example

Let G be a Lie group acting isometrically on a Riemannian manifold V. Then the set of orbits of G give rise to a singular Riemannian foliation.

Definition

We say that a point p in V is **regular** if the leaf through it has maximal dimension, otherwise it is **singular**.

Definition

A singular Riemannian foliation $\mathcal{F}$ in V is said to **admit sections** if there is a submanifold Σ through every regular point p in V meeting all orbits and always perpendicularly.

Remark

Singular Riemannian foliations admitting sections were first studied by Alexandrino and Töben.

Question

The orbit foliation of a polar action clearly admits sections.

When is the converse true?
In other words, when is a singular Riemannian foliation admitting sections the orbit foliation of a polar action?

Definition

A singular Riemannian foliation admitting sections in a round sphere S^m is called an **isoparametric family** and its regular leaves are called an **isoparametric submanifolds**.

Remark

An isoparametric submanifold determines its isoparametric family.

Remark

Isoparametric submanifolds were introduced by Terng in 1985 using a different definition.
There is a classical theory of isoparametric hypersurfaces in spheres due to Cartan (1939). There is an earlier work on such hypersurfaces in other space forms due to Somigliana, Segre, and Levi Civita.

Theorem (T. 1991)

Let $\mathcal{F}$ be an irreducible and full isoparametric foliation in a round sphere S^n such that the regular leaves have codimension at least two. Then $\mathcal{F}$ is the orbit foliation of a polar action.

Remark

The theorem is not true for codimension one isoparametric foliations: Inhomogeneous isoparametric hypersurfaces were found by Ozeki-Takeuchi and Ferus-Karcher-Münzner.

Remark

The proof of the theorem relies very strongly on results of Terng. In particular it uses her **Weyl group** of isoparametric foliations that generalize the one of polar actions. The proof also relies on Tits buildings.

After the spheres as ambient spaces we look at compact symmetric spaces.

Definition

An **equifocal family** in a compact symmetric space is a singular Riemannian foliation admitting sections which are flat. A regular leaf in such a family is said to be an **equifocal submanifold.**

Remark

An equifocal submanifold determines its equifocal family.

Remark

Equifocal families were introduced by Terng and myself using a different definition.

Remark

There is a structure theory of equifocal submanifolds that is analogous to the one of isoparametric submanifolds in spheres. Its most important invariant is an affine Weyl group acting on the universal covering covering space of a section.

Theorem (Christ 2002)

Let $\mathcal{F}$ be an irreducible equifocal family in a compact symmetric space. Then $\mathcal{F}$ is the orbit foliation of a polar action if the codimension of the principal leaves is at least two.

Remark

One cannot expect in general Riemannian manifolds that a singular Riemannian foliation admitting sections is an orbit foliation.

Conjecture

Let V be a compact **positively curved** *manifold with a singular Riemannian foliation $\mathcal{F}$ admitting sections and having leaves with codimension* **at least two**. *Then there is a homeomorphism h from V to a compact rank one symmetric space W such that $h(\mathcal{F})$ is the orbit foliation of a polar action.*

Curvature of Quotients

Let V be a Riemannian manifold on which a closed group G of isometries acts.

The quotient space $B = V/G$ is a metric space, but not a Riemannian manifold in general.

The set V_r of all points in V with principal isotropy group is open and dense in V. The quotient $B_r = V_r/G$ is a Riemannian manifold which is open and dense in B.

For a point $z \in B_r$, we denote by $\bar{\kappa}(z)$ the maximum of sectional curvatures of tangent planes at z.

Question

How does $\bar{\kappa}(z)$ behave as z approaches a point y in $B_{\mathrm{sing}} = B \setminus B_r$?

Theorem (Lytchak and T.)

Let V be a Riemannian manifold and let G be a closed group of isometries of M. Let $B = V/G$ be the quotient. Let $p \in V$ be a point with isotropy group G_p acting on the normal space ν_p of the orbit $Gp \subset V$.
Set $y = Kp \in B$.
Then the following are equivalent:

(i) $\limsup_{z \in B_r, z \to y} \bar{\kappa}(z) < \infty$;

(ii) $\limsup_{z \in B_r, z \to y} \bar{\kappa}(z) \cdot d^2(y, z) = 0$;

(iii) *The action of G_p on ν_p is polar;*

(iv) *A neighborhood of y in B is a smooth Riemannian orbifold.*