A new method of spherical expansions of zonal functions

Aleksander Strasburger

Warsaw University of Life Sciences — SGGW

based on a joint work with

A. Bezubik and A. Dąbrowska

Dedicated to Professor Sigurdur Helgason
with deep respect

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Notations and preliminaries on spherical harmonics

We adopt the following notations and definitions.

Denote by $(x|y)$ the Euclidean inner product of x, $y \in \mathbb{R}^d$, by $r^2 = (x|x)$ the (length of $x)^2$, and let $S = S^{d-1}$ be the unit sphere in $\mathbb{R}^d$. We consider only the case of $d \geq 3$ and set $\alpha = d/2 - 1$ $(\geq 1/2)$.

Let $\Delta f = \sum_{j=1}^{d} \partial^2 f/\partial x_j^2$ be the Laplacian in $\mathbb{R}^d$. Denote $\mathcal{P}^l = \mathcal{P}^l(\mathbb{R}^d)$ the space of polynomial functions on $\mathbb{R}^d$ homogeneous of degree l, and let $\mathcal{H}^l = \ker \Delta \cap \mathcal{P}^l$ be the space of harmonic and homogeneous of degree l functions. One knows that

$$\Delta : \mathcal{P}^l \longrightarrow \mathcal{P}^{l-2}$$

is surjective, thus

$$\dim \mathcal{H}^l = \dim \mathcal{P}^l - \dim \mathcal{P}^{l-2} = \frac{2(l+\alpha)\Gamma(2\alpha+l)}{\Gamma(l+1)\Gamma(2\alpha+1)}.$$

Both spaces $\mathcal{P}^l$ and $\mathcal{H}^l$ are invariant under the natural action on functions of the group $\mathbf{SO}(d)$ of rotations of $\mathbb{R}^d$ and it is moreover known that for each nonnegative integer l the space $\mathcal{H}^l$ is irreducible and the representations so obtained with different l are mutually inequivalent.

By the restriction to the unit sphere S each space $\mathcal{P}^l$ is mapped injectively into the space of continuous functions on S — the restriction of a polynomial in $\mathcal{H}^l$ to the unit sphere is called *a surface spherical harmonic* of order l. The space of surface spherical harmonics of order l is denoted by H^l — the natural action of $\mathbf{SO}(d)$ on it is equivalent to that on $\mathcal{H}^l$.

Let $d\sigma$ be the Euclidean surface measure on S (normalized so that $\int_S d\sigma = 1$) — by restriction to S each space $\mathcal{P}^l$ is injectively mapped into $L^2(S, d\sigma)$, so that

$$\langle\, P \mid Q \,\rangle = \int_S P(\xi)\overline{Q(\xi)}\, d\sigma(\xi), \tag{1}$$

provides an inner product in $\mathcal{P}^l$. The spaces of spherical harmonics of different orders regarded as subspaces of $L^2(S, d\sigma)$ are perpendicular, $H^l \perp H^k$ for $l \neq k$, and

$$L^2(S) = \bigoplus_{l=0}^{\infty} H^l,$$

with the orthogonal projections $f \mapsto P_l f$ onto H^l given by

$$P_l f(\xi) = \dim H^l \int_S Z^l(\eta,\, \xi) f(\eta) d\sigma(\eta). \tag{2}$$

Here the kernels $Z^l(\eta,\, \xi)$ themselves are, for any fixed $\xi \in S$, spherical harmonics of order l, whose explicit form will be given subsequently. The expansion

$$f(\xi) = \sum_{l=0}^{\infty} P_l f(\xi)$$

is called the *spherical harmonic expansion of* f.

As emphasized by R. Howe, a large part of the theory of spherical harmonics can be described in terms of an $\mathfrak{sl}_2$-triplet composed by $\mathbf{SO}(d)$-invariant operators $(1/4)\Delta$, r^2 (multiplication by r^2) and the shifted Euler operator $E = \sum_{i=1}^{d} x_i \partial_i + d/2$.

Acting on the polynomial algebra $\mathcal{P}(\mathbb{R}^d) = \bigoplus_{l=0}^{\infty} \mathcal{P}^l(\mathbb{R}^d)$ over $\mathbb{R}^d$ they satisfy the standard $\mathfrak{sl}_2$-commutation relations

$$[\frac{1}{4}\Delta,\, r^2] = E, \quad [E,\, \frac{1}{4}\Delta] = 2\frac{1}{4}\Delta, \quad [E,\, r^2] = -2r^2.$$

In particular each $\mathcal{P}^l$ is an eigenspace of E and the so-called canonical decomposition of homogeneous polynomials is nothing else as the spectral decomposition of the hermitian operator $r^2\Delta : \mathcal{P}^l \longrightarrow \mathcal{P}^l$. To describe it we introduce differential operators $\mathbf{E}^l_k$, $k = 0,\, 1,\, \ldots,\, [l/2]$, defined by

$$\mathbf{E}^l_k = \sum_{j=0}^{[l/2]-k} e^l{}_j(k) r^{2j} \Delta^{k+j},$$

where

$$e^l{}_j(k) = (-1)^j \frac{(\alpha + l - 2k)\Gamma(\alpha + l - 2k - j)}{4^{k+j} k! j! \Gamma(\alpha + l + 1 - k)},$$

with the property

$$\mathbf{E}^l_k : \mathcal{P}^l \longrightarrow \mathcal{H}^{l-2k}.$$

As usual, the notation $[m]$ signifies the integer part of a real number m.

We note that $\mathbf{E}^l_0$ is nothing else then the orthogonal projection $\mathbf{E}^l_0 : \mathcal{P}^l \longrightarrow \mathcal{H}^l$ (otherwise called the harmonic projection, cf. [12, 13]).

Theorem 1 (The canonical decomposition)
Any homogeneous polynomial $P \in \mathcal{P}^l$ decomposes as

$$P = \sum_{k=0}^{[l/2]} r^{2k} \mathbf{E}_k^l(P), \tag{3}$$

The operators $r^{2k}\mathbf{E}_k^l$ for $0 \le k \le [l/2]$ form a complete commutative family of orthogonal projections in $\mathcal{P}^l$ with respect to the inner product (1)*, so that* (3) *yields the orthogonal decomposition*

$$\mathcal{P}^l = \bigoplus_{k=0}^{[l/2]} r^{2k} \mathcal{H}^{l-2k}. \tag{4}$$

Moreover, the above decompositions are invariant with respect to the action of $K = \mathbf{SO}(d)$ in the respective spaces.

Thus any $P \in \mathcal{P}^l$ can be written as

$$P = \sum_{k=0}^{[l/2]} r^{2k} h_{l-2k}(P), \text{ with } h_{l-2k}(P) = \mathbf{E}_k^l(P) \in \mathcal{H}^{l-2k}. \tag{5}$$

The harmonic polynomial $h_l(P)$ of the highest degree in the right hand side of eqn. (5) is commonly called the harmonic projection of P and by analogy we shall call polynomials $h_{l-2k}(P)$ the harmonic components of P.

The form (3) of the canonical decomposition is operational and may be used for explicit computations. Before discussing its applications in the theory of harmonic expansions we present briefly its application for determination of Hermite polynomials in several variables. Recall, cf. Faraut [4], that given $P \in \mathcal{P}^l$ the Hermite polynomial H_P associated to P is given by

$$H_P(x) = (-2)^{-l} e^{r^2} P(\partial) e^{-r^2}, \qquad r^2 = |x|^2, \tag{6}$$

or, equivalently, by

$$H_P = \sum_{k=0}^{[l/2]} (-1)^k \frac{1}{2^{2k} k!} \Delta^k P.$$

Given $P \in \mathcal{P}^l$ with the canonical expansion (5), by applying the formulae for harmonic projections we obtain.

Proposition 1 *The Hermite polynomial H_P associated to P has the canonical decomposition of the form*

$$H_P = \sum_{k=0}^{[l/2]} (-1)^k k! L_k^{(\alpha+l-2k)}(r^2) h_{l-2k}(P). \tag{7}$$

where $L_k^{(\alpha+l-2k)}$ are the Laguerre polynomials given in terms of the confluent hypergeometric function ${}_1F_1$ by

$$L_k^{(\nu)}(x) = \frac{(\nu+1)_k}{k!} {}_1F_1(-n; \nu+1; x)$$

Proof:

$$P(\partial)e^{-r^2} = \left(\sum_{k=0}^{[l/2]} r^{2k} h_{l-2k}(P)\right)(\partial)e^{-r^2} \tag{8}$$

$$= \sum_{k=0}^{[l/2]} \Delta^k h_{l-2k}(P)(\partial)e^{-r^2} \tag{9}$$

$$= \sum_{k=0}^{[l/2]} (-2)^{l-2k} \Delta^k \left(e^{-r^2} h_{l-2k}(P)\right). \tag{10}$$

Now by virtue of [10, Lemma 3.2] we have

$$\Delta^k\big(e^{-r^2}h_{l-2k}(P)\big) = L_k(r^2)e^{-r^2}h_{l-2k}(P) \tag{11}$$

where

$$L_k(r^2) = \sum_{j=0}^{k}(-2)^{k+j}c^k_{l-2k,j}r^{2j}, \tag{12}$$

and the coefficients $c^k_{l-2k,j}$ are given by

$$c^k_{l-2k,j} = \begin{cases} \frac{1}{2^{k-j}(k-j)!}\lambda_{l-2k,j+1}\cdots\lambda_{l-2k,k}, & k > j \\ 1, & k = j \end{cases} \tag{13}$$

where $\lambda_{l-2k,p} = 2^2p(\alpha + l - 2k + p)$.

One may identify the polynomials L_k with the classical Laguerre polynomials by means of the following observations. Observe that for any natural m we have

$$\begin{aligned} c^k_{m,j} &= \frac{1}{2^{k-j}(k-j)!}\lambda_{m,j+1}\cdots\lambda_{m,k} \\ &= \frac{\lambda_{m,1}\cdots\lambda_{m,k}}{2^{k-j}(k-j)!\lambda_{m,1}\cdots\lambda_{m,j}} \\ &= 2^{k-j}(-1)^j(\alpha+1+m)_k\frac{(-k)_j}{j!(\alpha+1+m)_j}. \end{aligned}$$

Now recalling that the Laguerre polynomials are given by

$$L^{(\nu)}_k(x) = \frac{(\nu+1)_k}{k!}\sum_{j=0}^{k}\frac{(-k)_jx^j}{(\nu+1)_jj!}$$

one gets after some elementary manipulations

$$\Delta^k\big(e^{-r^2}h_{l-2k}(P)\big) = (-1)^k2^{2k}k!L^{(\alpha+l-2k)}_k(r^2)e^{-r^2}h_{l-2k}(P),$$

and the Proposition follows.

Analysis of zonal functions

We call a function f defined on the unit sphere S^{d-1} *a zonal function (relative to a point* $\eta \in S^{d-1}$*)* if it is invariant with respect to the isotropy group K_η of η. Any such function is in fact a function of one variable, namely the scalar product $(\xi \mid \eta)$, and as such can be written in the form $f(\xi) = \phi((\xi \mid \eta)),\ \xi \in S^{d-1}$, where the function ϕ defined on the unit interval $[-1, 1] \subset \mathbb{R}$ will be called the profile function of f.

Expansions of elementary zonal polynomials

Now we apply the above method to obtain the expansion of an elementary zonal polynomial $x \mapsto (x \mid \eta)^l$ with $\eta \in S^{d-1}$. Clearly

$$\Delta^k (x \mid \eta)^l = \frac{\Gamma(l+1)}{\Gamma(l+1-2k)} (x \mid \eta)^{l-2k} \tag{14}$$

and recaling the Gegenbauer polynomial $C_p^\alpha(\cdot)$ of degree p,

$$C_p^\alpha(t) = \sum_{j=0}^{[p/2]} (-1)^j \frac{\Gamma(\alpha+p-j)}{\Gamma(\alpha)\Gamma(j+1)\Gamma(p+1-2j)} (2t)^{p-2j}$$

with

$$C_p^\alpha(1) = \frac{\Gamma(2\alpha+p)}{p!\Gamma(2\alpha)} \tag{15}$$

we obtain

$$\mathbf{E}_k^l[(x \mid \eta)^l] = \sum_{j=0}^{[l/2]-k} e^l_{\ j}(k)\, r^{2(k+j)}\, \Delta^{k+j}(x \mid \eta)^l =$$

$$\frac{\Gamma(\alpha)\Gamma(l+1)(\alpha+l-2k)}{2^l k! \Gamma(\alpha+l+1-k)} r^l C_{l-2k}^\alpha\big((\xi \mid \eta)\big), \tag{16}$$

where for $x \neq 0$ we set $x = r\xi,\ \xi \in S^{d-1}$.

Substituting those expressions for $\mathbf{E}_k^l[(x \mid \eta)^l]$ into decomposition (3) we immediately obtain

Corollary 1.1 *For an arbitrary unit vector $\eta \in S^{d-1}$ and any nonnegative integer l the canonical decomposition of the polynomial $x \mapsto (x \mid \eta)^l$ is given by*

$$(\, x \mid \eta\,)^l = \frac{\Gamma(\alpha)\Gamma(l+1)}{2^l}|x|^l \sum_{k=0}^{[l/2]} \frac{(\alpha+l-2k)}{k!\Gamma(\alpha+l+1-k)} C_{l-2k}^{\alpha}\big((\,\xi \mid \eta\,)\big). \tag{17}$$

Since the group $\mathbf{SO}(d)$ acts irreducibly in H^l for each l there exists a unique, up to a scalar multiple, polynomial in H^l which is invariant under the action of the isotropy subgroup $K_\eta \simeq \mathbf{SO}(d-1)$ of a point $\eta \in S^{d-1}$. This polynomial, called the zonal polynomial of degree l with pole at η, defines the projector on H^l used in (2) and is given by

$$Z^l(\eta,\xi) = Z_\eta^l(\xi) = \big[C_l^\alpha(1)\big]^{-1} C_l^\alpha((\,\xi \mid \eta\,)). \tag{18}$$

The general form of an expansion

The spherical harmonic expansion for zonal functions reduces to

$$f(\xi) = \sum_{m=0}^{\infty} f_m Z_\eta^m(\xi), \text{ where } f_m = \dim H^m \int_{S^{d-1}} f(\xi) Z_\eta^m(\xi)\, d\sigma(\xi).$$

Classically the coefficients are expressed as integrals

$$f_m = \frac{(\alpha+m)\Gamma(\alpha)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_{-1}^{1} \phi(t) C_m^\alpha(t)(1-t^2)^{\alpha-1/2}\, dt$$

what reduces the problem to the expansion of the profile function ϕ with respect to the (orthogonal) system of Gegenbauer polynomials C_m^α. The following result shows how the coefficients f_m of the expansion can also be expressed in terms of the coefficients of the Taylor expansion of the profile function ϕ.

Theorem 2 *Let $f \in \mathcal{C}^\infty(S^{d-1})$ be a zonal function with pole at $\eta \in S^{d-1}$ and the profile function φ. If the Taylor expansion $\varphi(t) = \sum\limits_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} t^n$ of φ is absolutely convergent on the closed interval $[-1, 1]$, then $f(\xi)$ admits a uniformly convergent expansion*

$$f(\xi) = \sum_{m=0}^{\infty} d_m C_m^\alpha(1) Z_\eta^m(\xi), \tag{19}$$

where the coefficients d_m are given by

$$d_m = d_m(\varphi) = (\alpha + m)\Gamma(\alpha) \sum_{k=0}^{\infty} \frac{\varphi^{(m+2k)}(0)}{2^{m+2k} k! \Gamma(\alpha + m + k + 1)}. \tag{20}$$

Proof: After substituting $t = (\xi \,|\, \eta)$ in the Taylor series for φ and making use of (17) we obtain

$$\varphi((\xi \mid \eta)) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{\varphi^{(n)}(0)}{2^n} \frac{\Gamma(\alpha)(\alpha + n - 2k)}{k! \Gamma(\alpha + n - k + 1)} C_{n-2k}^\alpha((\xi \,|\, \eta)). \tag{21}$$

To show the order of summation may be changed, we renormalize the terms of the expansion with an aid of (15) and write

$$f(\xi) = \varphi((\xi \mid \eta)) =$$

$$\sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{[n/2]} \frac{n!(\alpha + n - 2k)\Gamma(\alpha)\Gamma(2\alpha + n - 2k)}{2^n k! \Gamma(\alpha + n + 1 - k)\Gamma(2\alpha)\Gamma(n + 1 - 2k)} \times$$

$$\left[C_{n-2k}^\alpha(1)\right]^{-1} C_{n-2k}^\alpha((\xi \,|\, \eta)).$$

Now, the estimate $|C_n^\alpha(t)| \le C_n^\alpha(1)$, that is valid for all $t \in [-1, 1]$, cf. eg. [1, Section 6.4], together with the identity

$$\frac{m!\Gamma(\alpha)}{2^m\Gamma(2\alpha)} \sum_{k=0}^{[m/2]} \frac{(\alpha+m-2k)\Gamma(2\alpha+m-2k)}{k!(m-2k)!\Gamma(\alpha+m+1-k)} = 1,$$

imply that the double series on the right hand side converges absolutely and uniformly over the unit sphere S^{d-1}. Setting $m = n - 2k$ one may rearrange the sum as follows

$$f(\xi) = \varphi((\xi\,|\,\eta)) =$$

$$\Gamma(\alpha) \sum_{m=0}^{\infty} (\alpha+m) C_m^\alpha((\xi\,|\,\eta)) \sum_{k=0}^{\infty} \frac{\varphi^{(m+2k)}(0)}{2^{m+2k} k! \Gamma(\alpha+m+k+1)}$$

which gives the asserted form of the expansion (19). □

This theorem enables us to obtain new derivation of the following (classical) results.

The plane wave expansion

The Bessel function of the first kind are defined by the series

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \left(\frac{t}{2}\right)^{2k} \tag{22}$$

and the so called spherical Bessel functions are $j_\nu(t) = \Gamma(\nu+1)\Big(t/2\Big)^{-\nu} J_\nu(t)$, where $\nu \in \mathbb{C}$ satisfies $\operatorname{Re}\nu > -1$.

Corollary 2.1 *For arbitrary unit vectors ξ, $\eta \in S^{d-1} \subset \mathbb{R}^d$ and $u \in \mathbb{R}_+$ the plane wave $e^{iu(\,\xi|\eta\,)}$ admits the following expansion*

$$e^{iu(\,\xi|\eta\,)} = \sum_{m=0}^{\infty} i^m \dim H^m \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+m+1)} \left(\frac{u}{2}\right)^m j_{\alpha+m}(u) Z_\eta^m(\xi).$$

The series converges absolutely for each fixed value of $u \in \mathbb{R}_+$ and uniformly with respect to ξ, $\eta \in S^{d-1}$.

The expansion of the Poisson kernel

The Poisson kernel for the unit ball in $\mathbb{R}^d$ is given by

$$P(r\eta;\xi) = \frac{1-r^2}{(1-2r(\xi \mid \eta)+r^2)^{d/2}}, \qquad \text{where } 0 \le r < 1. \tag{23}$$

For fixed r it is a zonal function with the pole η and from the Theorem 2 one obtains

Corollary 2.2

$$P(r\eta;\xi) = \sum_{m=0}^{\infty} \dim H^m r^m Z_\eta^m(\xi). \tag{24}$$

The expansion converges, for every fixed $0 \le r < 1$, *absolutely and uniformly with respect to* $\xi \in S^{d-1}$.

To prove this one first derives the expansion

$$\varphi_r\big((\xi\,|\,\eta)\big) = (1-2r(\xi\,|\,\eta)+r^2)^{-\alpha} = \sum_{m=0}^{\infty} r^m C_m^\alpha\big((\xi\,|\,\eta)\big), \tag{25}$$

that is known as the generating function for the Gegenbauer polynomials. On writing the Taylor series of $\varphi_r(t)$

$$\varphi_r(t) = (1-2tr+r^2)^{-\alpha} = \sum_{m=0}^{\infty} \frac{\Gamma(\alpha+m)}{m!\Gamma(\alpha)} \frac{(2r)^m}{(1+r^2)^{\alpha+m}}\, t^m,$$

it follows from (20) that the coefficients $d_m(\varphi_r)$ are given by

$$d_m(\varphi_r) = (\alpha+m)\sum_{k=0}^{\infty} \frac{\Gamma(\alpha+m+2k)}{k!\Gamma(\alpha+m+k+1)} r^{m+2k}(1+r^2)^{-\alpha-m-2k}.$$

By an application of the binomial formula one gets further

$$d_m(\varphi_r) = (\alpha+m)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^l \frac{\Gamma(\alpha+m+2k+l)}{k!l!\Gamma(\alpha+m+k+1)} r^{m+2(k+l)}$$
$$= (\alpha+m)\sum_{p=0}^{\infty}(-1)^p r^{m+2p}\sum_{k=0}^{p}\frac{(-1)^k\Gamma(\alpha+m+p+k)}{k!(p-k)!\Gamma(\alpha+m+k+1)}.$$

Remarkably enough, the summands corresponding to positive p vanish in virtue of an identity,

$$\sum_{k=0}^{p}(-1)^k \frac{\Gamma(\alpha+m+p+k)}{k!(p-k)!\Gamma(\alpha+m+k+1)} = 0, \qquad \text{for} \quad p>0, \quad (26)$$

so that $d_m(\varphi_r) = r^m$ for all m, as is needed for (25). To obtain the expansion of the Poisson kernel one can use a trick of Müller, cf. [8, Chapter 2.9] and apply an operator $1+(r/\alpha)\partial/\partial r$ to the generating function formula (25).

$$\frac{(1-r^2)}{(1-2r(\xi\,|\,\eta)+r^2)^{\alpha+1}} = \left(1+\frac{r}{\alpha}\frac{\partial}{\partial r}\right)\frac{1}{(1-2r(\xi\,|\,\eta)+r^2)^{\alpha}}$$
$$= \frac{1}{\alpha}\sum_{m=0}^{\infty}(\alpha+m)r^m C_m^{\alpha}\big((\xi\,|\,\eta)\big),$$

and since in virtue of (15) one has

$$\dim H^m = \frac{\alpha+m}{\alpha}C_m^{\alpha}(1),$$

the formula (24) is established.

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