”Derived Euler Numbers”

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Motivation

Let $\mathcal{C}$ denote a complex of finite dimensional vector spaces and linear maps

$$0 \longrightarrow E_0 \xrightarrow{\partial_0} E_1 \xrightarrow{\partial_1} E_2 \longrightarrow \cdots \longrightarrow E_r \xrightarrow{\partial_r} 0.$$

Let H_i be the i^{th} homology group.

The Euler number of $\mathcal{C}$ is given by

$$e(\{E_i, \partial_i\}) = \sum (-1)^i \dim H_i.$$

As the vector spaces are finite dimensional and they form a complex ($\partial_{i+1}\partial_i = 0$), one has also

$$e(\{E_i, \partial_i\}) = \sum (-1)^i \dim E_i.$$

Derived Euler Numbers

Set

$$p(t) = \sum_{i=0}^{r} \dim E_i \; t^i.$$

Then the Euler number is given by

$$e(\{E_i, \partial_i\}) = p(-1).$$

Define the derived Euler number $e'(\{E_i, \partial_i\})$ by

$$e'(\{E_i, \partial_i\}) = \sum (-1)^{i+1} i \dim E_i = p'(-1).$$

Notation

Let $\widetilde{X} = G/K$ be Hermitian globally symmetric of non-compact type.

Fix (τ, V) an irreducible, unitary representation of K subject to a minor restriction.

Let $\{Z_i\}$ be any orthonormal basis of $\mathfrak{p}_+$.

Let $e(\cdot)$ denote exterior multiplication on $\Lambda^*\mathfrak{p}_{\mathbb{C}}$.

Let (π, H_π) be an admissible representation of G with smooth vectors, H_π^∞.

Setting

The operator defined by

$$\overline{\partial}_\pi = \sum \pi(\overline{Z_i}) \otimes e(Z_i) \otimes I$$

is a linear map

$$\overline{\partial}_\pi : H_\pi^\infty \otimes \Lambda^q \mathfrak{p}_+ \otimes V \to H_\pi^\infty \otimes \Lambda^{q+1} \mathfrak{p}_+ \otimes V.$$

$(\overline{\partial}_\pi)^2 = 0$ and it commutes with K.

Get the $\overline{\partial}$ complex of finite dimensional vector spaces associated to $(\pi, V, \mathfrak{p}_-)$:

$$0 \to [H_\pi^\infty \otimes \Lambda^0 \mathfrak{p}_+ \otimes V]^K \xrightarrow{\overline{\partial}_\pi} [H_\pi^\infty \otimes \Lambda^1 \mathfrak{p}_+ \otimes V]^K \longrightarrow \cdots \xrightarrow{\overline{\partial}_\pi} [H_\pi^\infty \otimes \Lambda^n \mathfrak{p}_+ \otimes V]^K \to 0.$$

Let $\{H^{0,q}(\mathfrak{g}, K; \pi \otimes \tau)\}$ denote the homology groups.

Definition Set

$$e'(\pi, V, \mathfrak{p}_-) = \sum (-1)^{q+1} q \dim [H_\pi^\infty \otimes \Lambda^q \mathfrak{p}_+ \otimes V]^K$$

and call $e'(\pi, V, \mathfrak{p}_-)$ a spectral derived Euler number.

Problem Evaluate these derived Euler numbers for any irreducible representation (π, H_π).

•(π, H_π) discrete series - use Blattner's formula;

•(π, H_π) finite dimensional - K. Köhler;

•$(\pi, H_\pi) = \pi_{\xi,\nu} = \mathrm{Ind}_Q^G \xi \otimes e^\nu \otimes 1$,

if $\dim \mathfrak{a}_\mathfrak{q} \geq 2$ then $e'(\pi_{\xi,\nu}, V, \mathfrak{p}_-) = 0$

Maximal Parabolic Subalgebras

Koranyi, Wolf, Satake

$\{\text{max'l psa's}\} \Longleftrightarrow \{\kappa : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g} | \kappa \text{ al. cplx}\}.$

$\mathfrak{q}_\kappa = \mathfrak{m}_{\mathfrak{q}_\kappa} \oplus \mathfrak{a}_{\mathfrak{q}_\kappa} \oplus \mathfrak{n}_{\mathfrak{q}_\kappa} \Longleftrightarrow \qquad \kappa$

$X_\kappa = \kappa \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathfrak{a}_{\mathfrak{q}_\kappa} = \mathbb{R}X_\kappa$

$H_\kappa = \kappa \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$c_\kappa = \exp \frac{\pi i}{4} \kappa \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$C_\kappa = \operatorname{Ad} c_\kappa$,the κ-Cayley transform

The $\kappa(\mathfrak{sl}(2,\mathbb{R}))$ isotypic components

$$\mathfrak{g} = \mathfrak{g}^{[0]} \oplus \mathfrak{g}^{[1]} \oplus \mathfrak{g}^{[2]}$$

Maximal Parabolic Subalgebras, cont.

- $\mathfrak{n}_{\mathfrak{q}_\kappa} = \mathfrak{v} \oplus \mathfrak{u}$

  $\mathfrak{v} \Longleftrightarrow +1$ eigenspace ad X_κ

  $\mathfrak{u} \Longleftrightarrow +2$ eigenspace ad X_κ

- $\mathfrak{m}_{\mathfrak{q}_\kappa} = \mathfrak{m}_{\mathfrak{q}_\kappa}^{(1)} \oplus \mathfrak{m}_{\mathfrak{q}_\kappa}^{(2)}$

  $\mathfrak{m}_{\mathfrak{q}_\kappa}^{(1)} = \mathfrak{l}^{(1)} \oplus \mathfrak{g}_\kappa^{(1)} \Longleftrightarrow$ Herm, red

  $\mathfrak{m}_{\mathfrak{q}_\kappa}^{(2)} \Longleftrightarrow$ non-Herm, red

  $C_\kappa^{-1} : (\mathfrak{k}_\kappa^*)_{\mathbb{C}} \cong (\mathfrak{m}_{\mathfrak{q}_\kappa}^{(2)} \oplus \mathfrak{a}_{\mathfrak{q}_\kappa})_{\mathbb{C}}$

- $M_{Q_\kappa}^0 = M_{Q_\kappa}^{(1)} M_{Q_\kappa}^{(2)}$ and $M_{Q_\kappa}^+ = M_{Q_\kappa}^{(1)} M_{Q_\kappa}^{(2)} F_\kappa$.

Strategy

$$e'(\pi, V, \mathfrak{p}_-) = \sum (-1)^{q+1} q \dim[H_\pi^\infty \otimes \Lambda^q \mathfrak{p}_+ \otimes V]^K$$

$$e'(\pi_{\xi,\nu}, V, \mathfrak{p}_-) = \sum (-1)^{q+1} q \dim[W_\xi \otimes \Lambda^q \mathfrak{p}_+ \otimes V]^{K \cap M_Q^+}$$

$$= \sum \dim[\{(-1)^{q+1} q \Lambda^q \mathfrak{p}_+\} \otimes W_\xi \otimes V]^{K \cap M_Q^+}$$

We analyze $\{\sum (-1)^{q+1} q \Lambda^q \mathfrak{p}_+\}$ as virtual $K \cap M_Q^+$ module.

$$\mathfrak{p}_\pm = \mathfrak{p}_\pm^{[0]} \oplus \mathfrak{p}_\pm^{[1]} \oplus \mathfrak{p}_\pm^{[2]}$$

- K-W,S $\Longrightarrow \mathfrak{p}_\pm^{[0]} = \mathfrak{p}_\pm^{(1)}$

- K-W,S $\Rightarrow \mathfrak{p}_\pm^{[2]} = C_\kappa(\mathfrak{u}_\mathbb{C}) \cong \mathfrak{p}_\mathbb{C}^{(2)} \oplus \mathfrak{a}_{\mathfrak{q}_\kappa,\mathbb{C}}$

Set $L_\kappa =$ Centralizer$_K(H_\kappa)$. H_κ defines an almost complex structure on K/L_κ

$\mathfrak{l}_\kappa = \mathfrak{k} \cap \mathfrak{m}_{\mathfrak{q}_\kappa}^{(1)} \oplus \mathfrak{k}_\kappa^*$

$\mathfrak{n}_\kappa^c \Longleftrightarrow +i\, H_\kappa$ eigenspace in $\mathfrak{k}_\mathbb{C}$

$\overline{\mathfrak{n}}_\kappa^c \Longleftrightarrow -i\, H_\kappa$ eigenspace in $\mathfrak{k}_\mathbb{C}$

- K-W,S $\Longrightarrow \mathfrak{p}_+^{[1]} \simeq \overline{\mathfrak{n}}_\kappa^c$

Prop. As $K \cap M_{Q_\kappa}^{+}$ virtual module

$$\sum_{q=0}^{n} (-1)^{q+1} q \Lambda^{q} \mathfrak{p}_{+}$$

$$\begin{aligned} &\simeq (\Lambda^{\text{ev}} \mathfrak{p}_{+}^{(1)} - \Lambda^{\text{odd}} \mathfrak{p}_{+}^{(1)}) \\ &\otimes (\Lambda^{\text{ev}} \mathfrak{p}_{\mathbb{C}}^{(2)} - \Lambda^{\text{odd}} \mathfrak{p}_{\mathbb{C}}^{(2)}) \\ &\otimes \sum_{\ell=0}^{\dim \mathfrak{n}_{\kappa}^{c}} (-1)^{\ell+1} \Lambda^{\ell} \overline{\mathfrak{n}}_{\kappa}^{c} . \end{aligned}$$

Recall that

$$e'(\pi_{\xi,\nu}, V, \mathfrak{p}_{-}) = \sum \dim [\{(-1)^{q+1} q \Lambda^{q} \mathfrak{p}_{+}\} \otimes W_{\xi} \otimes V]^{K \cap M_{Q}^{+}}$$

Theorem (Kostant) Let V be an irreducible unitary representation of K with highest weight λ. Then as L_κ virtual modules

$$\sum(-1)^\ell \Lambda^\ell \overline{\mathfrak{n}}_\kappa^c \otimes V = \sum_{W_\kappa}(-1)^{\ell(w)} W_{w(\lambda+\rho_c)-\rho_c},$$

where W_μ is the irreducible L_κ module with highest weight μ.

Set $W_{\lambda_w} = W_{w(\lambda+\rho_c)-\rho_c}$.

As module for $K \cap M_{Q_\kappa}^{(1)} \times M_{Q_\kappa}^{(2)} F_\kappa$

$$W_{\lambda_w} = W_{\lambda_w^{(1)}} \otimes W_{\lambda_w^{(2)}}.$$

Prop. Let Q_κ be a maximal, proper, parabolic subgroup of G and let

$$\pi_{\xi,\nu} = \text{Ind}\,_{M_{Q_\kappa}^+ A_{Q_\kappa} N_{Q_\kappa}}^{G} (\xi \otimes e^\nu \otimes 1).$$

Then we have

$$e'(\pi_{\xi,\nu}, V, \mathfrak{p}_-) = \sum_{W_\kappa}(-1)^{\ell(w)+1} e(\xi^{(1)}, W_{\lambda_w^{(1)}}, \mathfrak{p}_-^{(1)}) e(\xi^{(2)}, W_{\lambda_w^{(2)}}, \mathfrak{p}_{\mathbb{C}}^{(2)}).$$

Euler Numbers

$e(\xi^{(1)}, W_{\lambda_w^{(1)}}, \mathfrak{p}_-^{(1)})$ is an Euler number for a $\overline{\partial}$-complex twisted with a vector bundle. For this we have

Lemma If $e(\xi^{(1)}, W_{\lambda_w^{(1)}}, \mathfrak{p}_-^{(1)}) \neq 0$ then

$$\xi^{(1)}(\Omega_{M_{Q\kappa}^{(1)}}) = \|\lambda_w^{(1)\vee} + \rho^{(1)}\|^2 - \|\rho^{(1)}\|^2$$

and the infinitesimal character $\Lambda_{\xi^{(1)}}$ of $\xi^{(1)}$ is

$$\Lambda_{\xi^{(1)}} = -w^{(1)}(\lambda_w^{(1)} + \rho^{(1)}).$$

Proof Convert $\overline{\partial}$-complex to spinors and Dirac, then use De George-Wallach.

Euler Numbers

$e(\xi^{(2)}, W_{\lambda_w^{(2)}}, \mathfrak{p}_{\mathbb{C}}^{(2)})$ is an Euler number for a de Rham complex twisted with a flat bundle. For this we have

Lemma If $e(\xi^{(2)}, W_{\lambda_w^{(2)}}, \mathfrak{p}_{\mathbb{C}}^{(2)}) \neq 0$ then

$$\xi_0^{(2)}(\Omega_{M_{Q\kappa}^{(2)}}) = \|\lambda_w^{(2)\vee} + \rho^{(2)}\|^2 - \|\rho^{(2)}\|^2$$

and the infinitesimal character of $\xi_0^{(2)}$, $\Lambda_{\xi_0^{(2)}}$, is given by

$$\Lambda_{\xi_0^{(2)}} = -w^{(2)}(\lambda_w^{(2)} + \rho^{(2)}).$$

Proof This is essentially a Theorem of Borel.

Characterization Theorem

Theorem Let ξ be a discrete series representation of $M_{Q_\kappa}^{+}$ and e^{ν} a character of A_{Q_κ}. Set $\pi_{\xi,\nu} = \mathrm{Ind}\,_{M_{Q_\kappa}^{+} A_{Q_\kappa} N_{Q_\kappa}}^{G}(\xi \otimes e^{\nu} \otimes 1)$. If

$$e'(\pi_{\xi,\nu}, V, \mathfrak{p}_-) \neq 0\,,$$

then for some $w \in W_\kappa$

$$\Lambda_\xi = (w^*(\lambda^*+\rho))\circ C_\kappa - 2\rho_n \circ C_\kappa + (\lambda_w\big|_{\mathbb{C}H_\kappa})\circ C_\kappa + \rho_{Q_\kappa}$$

and

$$\|\Lambda_\xi\|^2 - \|\lambda^*+\rho\|^2 = -\{(\lambda_w \circ C_\kappa + \rho_{Q_\kappa})(\widehat{X}_\kappa)\}^2 + 4\langle\lambda, \rho_n\rangle.$$

Finite Dimensional Rep'ns

$$\begin{aligned} e'(\pi, V, \mathfrak{p}_-) &= \sum (-1)^{q+1} q \,\mathrm{dim}[H_\pi^\infty \otimes \Lambda^q \mathfrak{p}_+ \otimes V]^K \\ &= \mathrm{dim}[\sum (-1)^{q+1} q \Lambda^q \mathfrak{p}_+ \otimes H_\pi^\infty \otimes V]^K \end{aligned}$$

Lemma Let ch denote the virtual character of an element of the representation ring $R(K)$. Then

$$ch(\sum (-1)^q q \Lambda_+^q) = ch(\sum (-1)^q \Lambda_+^q)(\frac{n}{2} + \frac{1}{2} \sum_{\Delta_n^+} coth \frac{\alpha}{2}).$$

Proposition Let (π, H_π) be a finite dimensional irreducible representation of G. If

$$e'(\pi, V, \mathfrak{p}_-) \neq 0\,,$$

then for some $\sigma \in W(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ and $w^* \in W_{\kappa_i}$ and $k \geq 1$

$$\sigma(\Lambda_\pi + \rho) = w^*(\lambda^* + \rho) - 2\rho_n + k\alpha_{\kappa_i}\,.$$

Orbits - Alg. Geometric

Refined Jacobson-Morozov

Open orbit - $\mathfrak{sl}_3$