A Paley-Wiener theorem for

Compact symmetric spaces.

To Sigurður: Happy birthday!

Reykjavik 18/8 - 07

[Joint work with Gestur Ólafsson]

Recall

G/K Riem. symm. space, non-compact type

1) Helgason (1966) – Gangolli (1971)

P.-W. thm: Characterizes image

$$\underbrace{\mathcal{F}}_{\text{spherical Fourier transform}}(\underbrace{C_c^\infty(G/K)^K}_{\text{left K-inv. functions}}) \subset \mathcal{O}(\mathfrak{a}_{\mathbb{C}}^*)$$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$

$\mathfrak{a} \subset \mathfrak{p}$ maximal abelian

2) Helgason (1973)

P-W. thm: Characterizes image

$$\underbrace{\mathcal{F}}_{\text{Fourier transform}}(C_c^\infty(G/K)) \subset \mathcal{O}(\mathfrak{a}_{\mathbb{C}}^*, C^\infty(K/M))$$

$M = K^{\mathfrak{a}}$

<u>Thm</u>: $\mathcal{F}\left(C_r^\infty(G/K)^K\right) = PW_r$ for $\forall r>0$

support in geodesic ball
centered at origin

$PW_r = \{ \varphi \in \mathcal{O}(\mathfrak{a}_{\mathbb{C}}^*) \mid$

a) $\forall m \; \exists C \quad |\varphi(\lambda)| \leq C(1+|\lambda|)^{-m} e^{r|\mathrm{Re}\,\lambda|}$

b) $\varphi(w\lambda) = \varphi(\lambda)$, $w \in W = N_K(\mathfrak{a})/M$.

Here:

$$\mathcal{F}f(\lambda) = \int_{G/K} f(x)\, \varphi_{-\lambda}(x)\, dx$$

spherical function

"New" proofs:

Spherical case:

Flensted-Jensen (1986):

Complex case (i.e. G complex group) easy

– simple formula for φ_λ

General case: Reduce to complex by

"Flensted-Jensen duality"

Uses result of Rais.

General case:

Torasso (1971):

General case follows from spherical case

Uses result of Kostant

Now

U/K Riem. symm. space, compact type

Want: Characterize $\mathcal{F}(C_r^\infty(U/K))$, $r>0$

First: spherical case, $C_r^\infty(U/K)^K$.

Notation

$\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{a} \subset \mathfrak{p}$ maximal abelian

$\Sigma = \{$ roots of $\mathfrak{a} \} \subset i\mathfrak{a}^*$ (purely imaginary)

Choose Σ^+

$\mathfrak{a} \subset \mathfrak{h} \subset \mathfrak{u}$ Cartan subalgebra, $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$.

Representation theory

Assume U simply connected:

Thm (Helgason)

$$\Lambda^+ = \{\mu \in \mathfrak{a}_{\mathbb{C}}^* \mid \forall \alpha \in \Sigma^+ : \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}^+ \} \subset i\mathfrak{a}^*$$

Then

$$\Lambda^+ \underset{\text{bijection}}{\longleftrightarrow} \{\text{irrep.s of } U \text{ with } K\text{-fixed vector} \neq 0\}$$

$$\mu \longleftrightarrow \delta_\mu$$

Put $\lambda_\mu \in \mathfrak{h}_{\mathbb{C}}^*$: $\lambda_\mu|_{\mathfrak{a}} = \mu$, $\lambda_\mu|_{\mathfrak{h}\cap\mathfrak{k}} = 0$

then λ_μ is the highest weight.

If not simply connected:

Replace by appropriate subset of Λ^+

Fourier transform

For $\mu \in \Lambda^+$ let $e_\mu \in V_\mu^K$, $e_\mu \neq 0$

K-fixed vector, $\|e_\mu\| = 1$

Def: $\psi_\mu(x) = (\delta_\mu(x) e_\mu, e_\mu)$, $\psi_\mu \in C^\infty(U/K)^K$

spherical function

$$\hat{f}(\mu) = \int_{U/K} f(x) \overline{\psi_\mu(x)} \, dx \quad , \quad \mu \in \Lambda^+$$

Fourier series :

$$f(x) = \sum_{\mu \in \Lambda^+} d(\mu) \hat{f}(\mu) \psi_\mu(x)$$

Thm (Sugiura) $f \in C^\infty \iff$

$$\forall m \, \exists C : \quad |\hat{f}(\mu)| \leq C (1 + |\mu|)^{-m}$$

Def: $PW_r = \{ \varphi \in \mathcal{O}(\mathfrak{a}_{\mathbb{C}}^*) \mid$

a) $\forall m \in \mathbb{N}\ \exists C: \quad |\varphi(\nu)| \leq C(1+|\nu|)^{-m} e^{r|\mathrm{Re}\,\nu|}$

b) $\varphi(w(\nu+\rho)-\rho) = \varphi(\nu) \quad , \quad \forall w \in W \quad \}$

Thm (Ólafsson + S.) $\quad \exists R>0 \ \forall 0<r<R:$

i) $\forall f \in C_r^\infty(U/K)^K \quad \exists \varphi \in PW_r: \quad \varphi|_{\Lambda^+} = \hat{f}$

ii) $\forall \varphi \in PW_r \quad \exists f \in C_r^\infty(U/K)^K: \quad \hat{f} = \varphi|_{\Lambda^+}$

iii) $\exists!$ in i) and ii)

So extension $\circ$ Fourier : $C_c^\infty(U/K)^K \to PW_r$

is an isomorphism

History

1) $U/K = S^1$. Elementary. Reduce to $\mathbb{R}$.

Similar for $(S^1)^n$

2) $U/K = S^2$. Beurling (unpublished)

3) General U/K of rank one. Koornwinder (1975)

4) Group case. $U \times U / \mathrm{diag}(U) \cong U$

Gonzalez (2001): P-W thm for class functions

Proof: Reduction to $(S^1)^n$ by Weyl character formula.

5) Even multiplicities of all roots.

Branson, Olafsson, Pasquale (2005)

6) Grassmannians. Camporesi (2006)

Proof

iii) Uniqueness.

In ii) the uniqueness is obvious : $\hat{f}$ determines f

In i) : $\varphi|_{\Lambda^+}$ determines φ.

Carlson's theorem:

$\varphi \in \mathcal{O}(\mathbb{C})$ is determined by $\varphi|_{\mathbb{Z}^+}$ if

$$\exists r < \pi : \quad |f(z)| \leq C e^{r|z|}$$

Note: π is important. Consider $\sin(\pi z)$.

i) Existence of holomorphic extension.

Recall:

G/K = non-compact dual form of U/K

(i.e. $G \subset G_{\mathbb{C}} \supset U$, $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p} \subset \mathfrak{g}_{\mathbb{C}}$)

Each ψ_μ extends to holomorphic function on $G_{\mathbb{C}}/K_{\mathbb{C}}$

– denote again ψ_μ.

Then $\psi_\mu\big|_{G/K}$ is a spherical function on G/K

– in fact $\psi_\mu\big|_{G/K} = \varphi_{\mu+\rho}$

Recall also:

Cartan decomposition: $U = K \cdot \exp\mathfrak{a} \cdot K$.

("spherical polar coordinates on U/K")

Def <u>Complex crown</u> of G/K:

$$Cr := \{ g \exp X \cdot x_0 \mid g \in G,\ X \in \mathfrak{a},\ \forall \alpha : |\alpha(X)| < \tfrac{\pi}{2} \}$$

$\uparrow$ recall $\mathfrak{a} \subset \mathfrak{p}$, $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$

$$\underset{\text{real}}{G/K \subset} Cr \underset{\text{open}}{\subset} G_{\mathbb{C}}/K_{\mathbb{C}}$$ (Akhiezer – Gindikin 1990)

<u>Thm</u> (Krötz – Stanton)

Every φ_λ, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, extends to a holo. function on Cr.

Note: B_r = centered ball in U/K

then $B_r \subset Cr$ if r small

Hence for $f \in C_r^\infty(U/K)^K$, r small, define:

$$\varphi(\nu) = \int_{B_r} f(x)\, \varphi_{\nu-\rho}(x)\, dx \quad , \quad \nu \in \mathfrak{a}^*_{\mathbb{C}}$$

<u>Thm</u> (Opdam)

Estimates.

ii) <u>Surjectivity in P-W theorem</u>

Given $\varphi \in PW_r$.

Sugiura gives $f \in C^\infty(U/K)^K$ with $\hat{f} = \varphi|_{\Lambda^+}$

Want: $\operatorname{supp} f \subset B_r$.

<u>Idea</u>: Reduce to group case (Gonzalez)
apply Rais (a la Flensted-Jensen)

<u>Rais</u>: $\exists\, p_1, \dots, p_\ell \in S(\mathfrak{a})^W$
$\exists\, \phi_1, \dots, \phi_\ell \in \mathcal{O}(\mathfrak{h}_{\mathbb{C}}^*)^{\tilde{W}}$

$$\varphi(\nu) = p_1(\nu)\phi_1(\nu) + \dots + p_\ell(\nu)\phi_\ell(\nu) \qquad (\nu \in \mathfrak{a}_{\mathbb{C}}^*)$$

$\tilde{W}$ = Weyl group of $\mathfrak{h}$

$N_{\tilde{W}}(\mathfrak{a}) \longrightarrow W$ is surjective

<u>Example</u>: $\mathfrak{a} = \mathfrak{h} = \mathbb{R}$, $W = \{e\}$, $\tilde{W} = \{\pm\}$

Every $\varphi \in \mathcal{O}(\mathbb{C})$ can be written

$$\varphi(z) = \phi_1(z) + z\,\phi_2(z)$$

even

Gonzalez :

$$\exists\ F_1, \dots, F_\ell \in C_r^\infty(U)^U \quad \text{with} \quad \hat{F} = \Phi_j$$

Put : $f_j(u) = \int_K F_j(uk)\,dk$, $f_j \in C_r^\infty(U/K)^K$

non-trivial geometric statement.

Choose $D_1, \dots, D_\ell \in \mathbb{D}(U/K)$ with $\gamma(D_j) = p_j$

$\mathcal{H}$-C. isomorphism

Put $f = \sum_j D_j f_j$. Then $\hat{f} = \varphi$.

(swept under rug : ρ-shift , dimensions)

General case (not K-invariant)

Define for $f \in C^\infty(U/K)$

$$\mathcal{F}f(\mu, kM) = (\delta_\mu(f)\, e_\mu\,,\ \delta_\mu(k)\, v_\mu)$$

$\mu \in \Lambda^+$, $k \in K$

K-fixed vector

highest weight vector

(definition introduced by Sherman, Stanton)

"<u>Thm</u>" (Ólafsson + S. – details not written down)

Similar result as that of Helgason '73.

<u>Proof</u>: Uses the idea of Torasso (involves Kostant).

Reduction to K-types.

Apply $\mathcal{U}(\mathfrak{g})$ to go from trivial K-type to a given K-type:

K-fixed vector is cyclic in the spherical pr. ser for λ in positive chamber – use W.