SYMMETRIC SPACES AND THE KASHIWARA-VERGNE METHOD

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til Sigurðar Helgasonar með þökkum, vinsemd og virðingu

Introduction

Let G/K denote a Riemannian symmetric space of the non-compact type.

> *"The action of the G-invariant differential operators on G/K on the radial functions on G/K is isomorphic with the action of certain differential operators with constant coefficients. The isomorphism in question is used in Harish-Chandra's work on the Fourier analysis on G and is related to the Radon transform on G/K.*
>
> *For the case when G is complex a more direct isomorphism of this type is given, again as a consequence of results of Harish-Chandra."*

This is a quote from S. Helgason [8] *Fundamental solutions of invariant differential operators on symmetric spaces* (1964), one of the first mathematical papers I studied. Those two results were fascinating to me, they still are and they motivated my everlasting interest in invariant differential operators as well as Radon transforms. Yet I was dreaming simpler proofs could be given, without relying on Harish-Chandra's deep study of semisimple Lie groups...

Then, in the fall of 1977, came a preprint by Kashiwara and Vergne [11] *The Campbell-Hausdorff formula and invariant hyperfunctions*, showing that similar results could be obtained - for solvable Lie groups at least - only by means of "elementary" (but very clever) computations with the exponential mapping and the Campbell-Hausdorff formula.

After briefly recalling this method for Lie groups I will give an overview of its extension to general symmetric spaces, including several recent results.

1. The Kashiwara-Vergne method for Lie groups

Let G be a Lie group with Lie algebra $\mathfrak{g}$. The Campbell-Hausdorff formula

$$\log(\exp X \exp Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots \tag{1}$$

expresses (locally) the group law of G in the exponential chart $\exp : \mathfrak{g} \to G$ as a convergent series of iterated Lie brackets. Kashiwara and Vergne [11] stated the following

Conjecture 1 *The formula can be written as*

$$Z := \log(\exp Y \exp X) = X + Y + (e^{-\operatorname{ad} X} - 1)F(X,Y) + (1 - e^{\operatorname{ad} Y})G(X,Y) \quad (2)$$

where $F, G : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *are given by convergent power series near* $(0,0)$ *and satisfy the trace condition*

$$\operatorname{tr}_{\mathfrak{g}} (\operatorname{ad} X \cdot \partial_X F + \operatorname{ad} Y \cdot \partial_Y G) = \frac{1}{2} \operatorname{tr}_{\mathfrak{g}} \left(\frac{\operatorname{ad} X}{e^{\operatorname{ad} X} - 1} + \frac{\operatorname{ad} Y}{e^{\operatorname{ad} Y} - 1} - \frac{\operatorname{ad} Z}{e^{\operatorname{ad} Z} - 1} - 1 \right) . \quad (3)$$

Formula (2) is rather easily obtained: roughly speaking one can group together all brackets $[X, ...]$ resp. $[Y, ...]$ in the right-hand side of (1), though not in a unique way. But (3) is far from obvious!

Let $j(X) = \det_{\mathfrak{g}} \left(\frac{1 - e^{-\operatorname{ad} X}}{\operatorname{ad} X} \right)$, the jacobian of exp. Working on the domain of the chart exp, we associate to a function f on $\mathfrak{g}$ the function $\widetilde{f}$ on G such that

$$f(X) = j(X)^{1/2} \widetilde{f}(\exp X) \ .$$

This correspondence extends to distributions u (on $\mathfrak{g}$) and $\widetilde{u}$ (on G) with

$$< \widetilde{u}, \widetilde{f} > = < u, f >$$

for any test function f on $\mathfrak{g}$.

A distribution u on $\mathfrak{g}$ is said to be *invariant* if $< u(X), f(\operatorname{Ad} g(X)) > = < u(X), f(X) >$ for all $g \in G$, $f \in \mathcal{D}(\mathfrak{g})$.

Theorem 2 *(Kashiwara-Vergne) If the conjecture is true, then*

$$\widetilde{u} *_G \widetilde{v} = \widetilde{u *_{\mathfrak{g}} v}$$

for any invariant distributions u, v *such that the convolutions make sense.*

Taking v supported at the origin it follows that bi-invariant differential operators on G correspond to constant coefficients differential operators on the Lie algebra by means of $\widetilde{\ }$.

Theorem 3 *(Kashiwara-Vergne) The conjecture is true for solvable Lie algebras.*

The conjecture is also true for $\mathfrak{g} = sl(2, \mathbb{R})$ (R. 1981 [14], a lengthy and boring computation). What about general Lie algebras?

What can you do when you fail to prove a difficult conjecture? Either you give up or you extend the conjecture. I did both: I extended it to symmetric spaces, then I gave up. Decisive progress was made in the last decade fortunately:

- true for any quadratic Lie algebra (Vergne 1999 [20], Alekseev and Meinrenken 2002 [1])

- true for all Lie algebras (Andler, Sahi and Torossian 2001 [4], Torossian 2002 [18], Alekseev and Meinrenken 2006 [2]). See the June 2007 Bourbaki seminar [19] by Torossian for a nice survey. Those works rely on M. Kontsevich's fundamental paper [12] on quantization.

Let us also mention two interesting recent works by Alekseev and Petracci (2006) [3] and by Burgunder (2006) [5], discussing the uniqueness of functions F, G in the conjecture. The former paper proves the uniqueness, up to order one in Y, of F and G satisfying (2) and (3) and the natural symmetry $G(X,Y) = F(-Y,-X)$. The latter gives all solutions F, G of (2) alone.

2. e-functions of symmetric spaces

2.a. Definition

Let G/H denote a general symmetric space and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ the decomposition of the Lie algebra given by the symmetry. Here $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{s}$ identifies to the tangent space to G/H et the origin. Let $\exp : \mathfrak{g} \to G$ and $\operatorname{Exp} : \mathfrak{s} \to G/H$ be the respective exponential mappings. *We shall always work on neighborhoods of the origin where they are diffeomorphisms.* To simplify the exposition this will not be repeated in the sequel.

Let

$$J(X) = \det_{\mathfrak{s}} \left(\frac{\operatorname{sh} \operatorname{ad} X}{\operatorname{ad} X} \right) \ , \ X \in \mathfrak{s} \ ,$$

be the Jacobian of Exp. As before we relate a function f, resp. a distribution u, on $\mathfrak{s}$ to $\widetilde{f}$, resp. $\widetilde{u}$, on G/H by

$$f(X) = J(X)^{1/2} \widetilde{f}(\operatorname{Exp} X) \ , \ < \widetilde{u}, \widetilde{f} > = < u, f > \ .$$

The distribution u on $\mathfrak{s}$ is said to be H-*invariant* if $< u(X), f(\operatorname{Ad} h(X)) > = < u(X), f(X) >$ for all $h \in H$, $f \in \mathcal{D}(\mathfrak{s})$. The nice convolution formula of the group case no longer holds for general symmetric spaces. One must consider "twisted convolutions" instead and give the following

Definition 4 *An analytic function* $e : \mathfrak{s} \times \mathfrak{s} \to \mathbb{R}$ *is called* ***an*** e***-function of*** G/H *if* $e(h \cdot X, h \cdot Y) = e(X, Y)$ *for all* $h \in H$, $X, Y \in \mathfrak{s}$ *(adjoint action) and, for all* H*-invariant distributions* u, v *and all test functions* f *on* $\mathfrak{s}$,

$$< \widetilde{u} *_{G/H} \widetilde{v}, \widetilde{f} > = < u(X) \otimes v(Y), e(X, Y) f(X + Y) > \ . \tag{4}$$

The symmetric space is said to be ***special*** *if it admits an e-function which is identically* 1.

The convolution on the left is defined by

$$< \widetilde{u} *_{G/H} \widetilde{v}, \widetilde{f} > = < \widetilde{u}(xH), < \widetilde{v}(yH), \widetilde{f}(xyH) >>$$

where x, y are variables in G (see [15] §1 for more details).

Example. Let G/K be a *rank one* Riemannian symmetric space of the non-compact type, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as usual, and let p, q be the respective multiplicities of the roots $\alpha, 2\alpha$. The dimension of G/K is $n = p + q + 1$. Then an e-function of G/K is given by (Flensted-Jensen & R., unpublished)

$$e(X, Y) = \left(4 \frac{x}{\operatorname{sh} x} \cdot \frac{y}{\operatorname{sh} y} \cdot \frac{z}{\operatorname{sh} z} \cdot \frac{\operatorname{ch}(x+y) - \operatorname{ch} z}{(x+y)^2 - z^2} \cdot \frac{\operatorname{ch} z - \operatorname{ch}(x-y)}{z^2 - (x-y)^2} \right)^{(n-3)/2} \times$$
$$\times \ {}_2F_1 \left(1 - \frac{q}{2}, \frac{q}{2}; \frac{n-1}{2}; \frac{(\operatorname{ch}(x+y) - \operatorname{ch} z)(\operatorname{ch} z - \operatorname{ch}(x-y))}{4 \operatorname{ch} x \operatorname{ch} y \operatorname{ch} z} \right) \tag{5}$$

where $X, Y \in \mathfrak{p}$, $x = \|X\| = (-B(X, \theta X)/2(p + 4q))^{1/2}$ (normalized Killing form), $y = \|Y\|$, $z = \|X + Y\|$. The result comes out of manipulations of various integral formulas. Note that the hypergeometric factor ${}_2F_1$ is 1 if $q = 0$ i.e. $G/K = H^n(\mathbb{R})$, and this space is special if $n = 3$.

2.b. Application to invariant differential operators

Let $p \in S(\mathfrak{s})^{\mathfrak{h}}$ be an $\mathfrak{h}$-invariant element in the symmetric algebra of $\mathfrak{s}$, identified to an H-invariant differential operator $p(\partial)$ on $\mathfrak{s}$ with constant coefficients. The distribution $v = {}^t p\delta$ (where ^{t} means transpose and δ is the Dirac measure at the origin of $\mathfrak{s}$) is then H-invariant, supported at the origin. Applying (4) we obtain, for any H-invariant distribution u on $\mathfrak{s}$,

$$< \widetilde{u} * \left({}^t p\delta\right)^{\sim}, \widetilde{f} >=< u(X) \otimes^t p\delta(Y), e(X,Y)f(X+Y) > \ .$$

The left-hand side is $< {}^t\widetilde{p}\widetilde{u}, \widetilde{f} >$, where $\widetilde{p}$ is the G-invariant differential operator on G/H defined by

$$\widetilde{p}\varphi(\operatorname{Exp} X) = p(\partial_Y)\left(J(Y)^{1/2}\varphi(\exp X \cdot \operatorname{Exp} Y)\right)\Big|_{Y=0} \ , \varphi \in C^\infty(G/H) \ .$$

In the chart Exp, the operator $\widetilde{p}$ acting on H-invariant distributions is then expressed by

$$ {}^t\widetilde{p}\widetilde{u} = \left({}^t p_e(X, \partial_X)u\right)^{\sim} \qquad (6)$$

where $p_e(X, \partial_X)$ is the H-invariant differential operator on $\mathfrak{s}$ defined by

$$p_e(X, \partial_X)f(X) = p(\partial_Y)\left(e(X,Y)f(X+Y)\right)|_{Y=0} \ ,$$

with symbol

$$p_e(X,\xi) = \sum_\alpha \frac{1}{\alpha!}\partial_Y^\alpha e(X,0) \cdot p^{(\alpha)}(\xi) \ , \ X \in \mathfrak{s} \ , \ \xi \in \mathfrak{s}^*$$

(finite sum).

Example 1. Let G/H be (pseudo-)Riemannian, with Laplace operator $L_{G/H}$, and let $p = L_{\mathfrak{s}}$ be the Laplacian of $\mathfrak{s}$. For any H-invariant function (or distribution) u on $\mathfrak{s}$ formula (6) reduces to

$$L_{G/H}\widetilde{u} = \left(L_{\mathfrak{s}}u - J^{-1/2}L_{\mathfrak{s}}J^{1/2} \cdot u\right)^{\sim}$$

(Helgason, 1972; see [9] p. 273). In this simple example only second derivatives of e are needed, and they can be expressed by means of $J^{1/2}$.

Example 2. If G/H is special ($e = 1$) then $p_e(X, \partial_X) = p(\partial_X)$.

The map $p \mapsto \widetilde{p}$ is thus a linear isomorphism of $S(\mathfrak{s})^{\mathfrak{h}}$ onto $\mathbb{D}(G/H)$, the algebra of G-invariant differential operators on the symmetric space, but not in general an isomorphism of algebras. Instead we have the following corollary of (6) :

Theorem 5 *(R. 1991 [16]) Let $S(\mathfrak{s})^{\mathfrak{h}}$ be equipped with the product $\times$ defined by*

$$(p \times q)(\xi) = e(\partial_\xi, \partial_\eta)p(\xi)q(\eta)|_{\xi=\eta} = \sum_{\alpha,\beta} \frac{1}{\alpha!\beta!}\partial_X^\alpha\partial_Y^\beta e(0,0)\ p^{(\alpha)}(\xi)q^{(\beta)}(\xi) \ .$$

Then $p \mapsto \widetilde{p}$ is an isomorphism of algebras of $(S(\mathfrak{s})^{\mathfrak{h}}, \times)$ onto $(\mathbb{D}(G/H), \circ)$, i.e. $\widetilde{p \times q} = \widetilde{p} \circ \widetilde{q}$.

Of course $\times$ is the usual product in the symmetric algebra if G/H is special ($e = 1$).

2.c. A general construction of e-functions

The analogue of $\log(\exp X \exp Y)$ above is now $Z(X,Y) \in \mathfrak{s}$ defined by

$$\operatorname{Exp} Z(X,Y) = \exp X \cdot \operatorname{Exp} Y \ , \ X,Y \in \mathfrak{s} \ .$$

Thus Z expresses the natural action of G on G/H read in exponential coordinates. It is readily checked by means of the symmetry that $2Z(X,Y) = \log(\exp X \exp 2Y \exp X)$; thus Z is given by a variant of the classical Campbell-Hausdorff formula. Working on this expression the following can be shown.

Theorem 6 *(R. 1986 [15]) There exist two analytic maps $a, b : \mathfrak{s} \times \mathfrak{s} \to H$ such that*

$$(X,Y) \longmapsto \Phi(X,Y) = (a(X,Y)\cdot X, b(X,Y)\cdot Y)$$

(adjoint action of H on $\mathfrak{s}$) is a diffeomorphism of $\mathfrak{s} \times \mathfrak{s}$ onto itself such that

$$Z \circ \Phi(X,Y) = X + Y \ .$$

Thus Φ transforms Z into its analogue for the flat symmetric space $\mathfrak{s}$. It is obtained as the value for $t = 1$ of a one-parameter family of diffeomorphisms Φ_t solving differential equations related to the evolution for $0 \le t \le 1$ of $Z_t(X,Y) = t^{-1}Z(tX,tY)$, $Z_0(X,Y) = X + Y$. In others words one works with *contractions of the symmetric space G/H* into its tangent space $\mathfrak{s}$.

For the sake of simplicity let us now *assume G/H admits a G-invariant measure*, a non-essential assumption so as to avoid some modular factors in the formulas.

Changing variables by means of the diffeomorphism Φ leads to

Theorem 7 *(R. 1990 [16]) The formula*

$$e(X,Y) = \left(\frac{J(X)J(Y)}{J(X+Y)}\right)^{1/2} \det_{\mathsf{s}\times\mathsf{s}} D\Phi(X,Y) \tag{7}$$

gives an e-function of G/H, strictly positive and such that

$$e(X,Y) = e(-X,-Y) = e(Y,X)$$

Let $b = B_{\mathsf{g}} - 2B_{\mathsf{h}}$ where B_{g}, B_{h} are the Killing forms of $\mathfrak{g}$ and $\mathfrak{h}$. Then

$$e(X,Y) = 1 - \frac{1}{240} b([X,Y],[X,Y]) + \frac{1}{1512} b([X,Y],u[X,Y]) + \cdots$$

with $u = (\operatorname{ad} X)^2 + \operatorname{ad} X \operatorname{ad} Y + (\operatorname{ad} Y)^2$.

It is difficult to compute the Jacobian $\det D\Phi$ explicitly. Nevertheless Theorem 7 implies the following:

- (R. 1990 [16]) G/H is special if G is solvable.
- $G_{\mathbb{C}}/G_{\mathbb{R}}$ and $G \times G/\mathit{diagonal}$ are special for any G (this result relies on the Kashiwara-Vergne conjecture for G).

- (R. 2007) For $p \in S(\mathfrak{s})^{\mathfrak{h}}$ let $p_e(X, \partial_X)f(X) = p(\partial_Y)\left(e(X,Y)f(X+Y)\right)|_{Y=0}$ as above. Then, for any $f \in C^\infty(\mathfrak{s})$,

$$\widetilde{p}\widetilde{f} = (p_e f + rf)^{\sim}, \tag{8}$$

where r is a differential operator on $\mathfrak{s}$ belonging to the left ideal generated by the adjoint vector fields

$$\zeta_V f(X) = \frac{d}{dt} f((\exp -tV)\cdot X)\bigg|_{t=0}, \ V \in \mathfrak{h}\ ,\ X \in \mathfrak{s}\ .$$

In particular $rf = 0$ whenever f is an H-invariant function. This extends to symmetric spaces Proposition 4.2 of [11] in the case of Lie groups.

Remark. In view of the above factors $J^{1/2}$ it would be more natural to work with half-densities rather than functions on the symmetric space. The theory actually extends to line bundles over G/H ([17]).

2.d. A new construction of e-functions

Adapting Kontsevich's quantization to symmetric spaces ($\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ as above) Cattaneo and Torossian ([6], §3) define a function $E(X,Y)$ of $X, Y \in \mathfrak{s}$ and a $*$-product

$$(p * q)(\xi) = E(\partial_\xi, \partial_\eta)(p(\xi)q(\eta))|_{\xi=\eta}\ ,$$

with $p, q \in S(\mathfrak{s})$, $\xi \in \mathfrak{s}^*$. When restricted to H-invariant elements p, q, this product is associative and commutative.

They show that

$$\log E(X,Y) = \sum_\Gamma w_\Gamma \left(\operatorname{tr}_{\mathfrak{s}}(x_1 x_2 \cdots x_n) + (-1)^n \operatorname{tr}_{\mathfrak{h}}(x_n \cdots x_2 x_1)\right)$$

where $x_i = \operatorname{ad} X_i$ and each X_i consists of iterated brackets of X and Y. The sum runs over a family of graphs Γ adapted from Kontsevich's diagrams, and the w_Γ are certain coefficients. The main properties of E can thus be read off "easily" from the graphs.

In particular $E(X,Y) = 1$ if $\mathfrak{g}$ solvable, or for a quadratic Lie algebra viewed as a symmetric pair.

Cattaneo and Torossian also obtain an explicit expression, by means of $*$-products, of invariant differential operators on G/H written in exponential coordinates ([6], §4).

Besides the product $*$ above, restricted to H-invariant $p, q \in S(\mathfrak{s})$, coincides with $\times$ in Theorem 5

$$\widetilde{p * q} = \widetilde{p \times q} = \widetilde{p} \circ \widetilde{q}\ ,$$

as differential operators acting on H-invariant functions.

Thus the function $E(X,Y)$ is very similar to $e(X,Y)$ in Theorem 7 but the proofs of its main properties are much easier (inspection of graphs instead of difficult identities in non-commutative algebra). It is reasonable to conjecture that $E(X,Y)$ and $e(X,Y)$ are (essentially) the same function; see [6] §4.2.2 for a more precise statement. But this remains to be proved.

3. Back to the introduction

For special symmetric spaces the problems mentioned in the introduction are solved by the map $\tilde{}$. For more general spaces one should expect it can be done by means of an e-function. We now explain this in the case of a Riemannian symmetric space G/K, using the classical semi-simple notations as in Helgason's books [9][10].

4.a. We recall that Harish-Chandra's spherical transform $F_{G/K}$ is related to the Abel transform $\mathcal{A} : \mathcal{D}(G/K)^K \to \mathcal{D}(\mathfrak{a})^W$ by

$$\int_{G/K} f(x)\varphi_\lambda(x)\, dx = F_{G/K}f(\lambda) = F_{\mathfrak{a}}\mathcal{A}f(\lambda)\ ,\ \lambda \in \mathfrak{a}^*\ , \tag{9}$$

where $F_{\mathfrak{a}}f(\lambda) = \int_{\mathfrak{a}} f(H)e^{i<\lambda,H>}dH$ is the classical Fourier transform on the vector space $\mathfrak{a}$.

By the Weyl group invariance $\varphi_{w\lambda} = \varphi_\lambda$ and a suitable version of Chevalley's restriction theorem ([9] p. 468-469) this W-invariant equality in $\lambda \in \mathfrak{a}^*$ extends to a K-invariant equality in $\xi \in \mathfrak{p}^*$:

$$\int_{G/K} f(x)\varphi_\xi(x)\, dx = F_{G/K}f(\xi) = F_{\mathfrak{p}}\mathcal{T}f(\xi)\ ,\ \xi \in \mathfrak{p}^*\ , \tag{10}$$

where $F_{\mathfrak{p}}f(\xi) = \int_{\mathfrak{p}} f(X)e^{i<\xi,X>}dX$ is the Fourier transform on $\mathfrak{p}$. This defines an **operator** $\mathcal{T} : \mathcal{D}(G/K)^K \to \mathcal{D}(\mathfrak{p})^K$ which transforms convolutions of K-invariant functions on G/K into the abelian convolution on $\mathfrak{p}$ and G-invariant differential operators on G/K into K-invariant differential operators on $\mathfrak{p}$ with constant coefficients.

Let $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{q}$ (orthogonal sum) and, for $\psi \in \mathcal{D}(\mathfrak{p})^K$, $H \in \mathfrak{a}$, $\mathcal{A}_0\psi(H) = \int_{\mathfrak{q}} \psi(H+Y)\, dY$ (flat Abel transform; see [10] chap. IV §5). Then $F_{\mathfrak{p}}\psi(\lambda) = F_{\mathfrak{a}}\mathcal{A}_0\psi(\lambda)$, $\lambda \in \mathfrak{a}^*$; this equality is the flat analogue of (9). Comparing (9) and (10) we have $F_{\mathfrak{a}}\mathcal{A}f(\lambda) = F_{\mathfrak{p}}\mathcal{T}f(\lambda) = F_{\mathfrak{a}}\mathcal{A}_0\mathcal{T}f(\lambda)$ therefore

$$\mathcal{T} = \mathcal{A}_0^{-1} \circ \mathcal{A}\ .$$

4.b. This nice operator $\mathcal{T}$ can be related to e-functions in the following way.

Conjecture 8 *Let e be the function introduced in Theorem 7. For $Y \in \mathfrak{p}'$ (a regular element in $\mathfrak{p}$) the following limit*

$$e_\infty(X,Y) = \lim_{t\to+\infty} e(X,tY)$$

exists, with uniform convergence when X runs in an arbitrary compact subset of $\mathfrak{p}$.

I am grateful to M. Flensted-Jensen for suggesting to consider this limit. Clearly, for $k \in K$ and $t > 0$,

$$e_\infty(X,tY) = e_\infty(X,Y) = e_\infty(k\cdot X, k\cdot Y)\ .$$

Examples. As an easy consequence of (5), Conjecture 8 is true for rank one spaces and

$$e_\infty(X,Y) = \left(2\frac{x}{\operatorname{sh} x}\frac{\operatorname{ch} x - \operatorname{ch} s}{x^2 - s^2}\right)^{(n-3)/2} {}_2F_1\left(1 - \frac{q}{2}, \frac{q}{2}; \frac{n-1}{2}; \frac{\operatorname{ch} x - \operatorname{ch} s}{2\operatorname{ch} x}\right) \tag{11}$$

with $X, Y \in \mathfrak{p}$, $\|X\| = x$, $\|Y\| = 1$ and $s = X\cdot Y$ (the K-invariant scalar product on $\mathfrak{p}$ corresponding to the norm $\|.\|$).

Besides Conjecture 8 is obviously true when G/K is special ($e = 1$), i.e. when G admits a complex structure.

Theorem 9 *Assume Conjecture 8. Then the spherical functions (extended to $\xi \in \mathfrak{p}^{*\prime}$, identified with a regular element of $\mathfrak{p}$ by duality) are*

$$\varphi_\xi(\operatorname{Exp} X) = J(X)^{-1/2} \int_K e^{i<\xi, k\cdot X>} e_\infty(k \cdot X, \xi) \; dk \; , \; X \in \mathfrak{p} \; , \tag{12}$$

and the above operator $\mathcal{T}$ is given by the oscillatory integral

$$\mathcal{T}\widetilde{u}(Y) = \int_{\mathsf{p}\times\mathsf{p}^{*\prime}} e^{i<\xi, X-Y>} e_\infty(X,\xi) u(X) \; dX d\xi \; , \; u \in \mathcal{D}(\mathfrak{p})^K, \; Y \in \mathfrak{p} \; .$$

Thus, up to the map $u \mapsto \widetilde{u}$, $\mathcal{T}$ is a **pseudo-differential operator** of order 0 on $\mathfrak{p}$ defined by the e-function. If G admits a complex structure it boils down to $\mathcal{T}\widetilde{u} = u$.

Remarks. An expression of spherical functions similar to (12) has been obtained by Duistermaat (1983) [7] by means of a diffeomorphism of the boundary K/M transforming the Iwasawa projection into the orthogonal projection $\mathfrak{p} \to \mathfrak{a}$. Theorem 6 draws inspiration from his construction, which in turn appears as some boundary limit of ours in the spirit of Conjecture 8. A proof of this conjecture might follow from unifying Duistermaat's diffeomorphism and our Φ in Theorem 6.

For rank one spaces (11) and (12) give back a formula for spherical functions already proved by Koornwinder (1975) [13].

References

[1] Alekseev, A. and Meinrenken, E., *Poisson geometry and the Kashiwara-Vergne conjecture*, C.R. Acad. Sci. Paris, Série I, 335 (2002), 723-728.

[2] Alekseev, A. and Meinrenken, E., *On the Kashiwara-Vergne conjecture*, Invent. Math. 164 (2006), 615-634.

[3] Alekseev, A. and Petracci, E., *Low order terms of the Campbell-Hausdorff series and the Kashiwara-Vergne conjecture*, J. Lie Theory 16 (2006), 531-538; arXiv:math.QA/0508077.

[4] Andler, M., Sahi, S. and Torossian, C., *Convolution of invariant distributions: proof of the Kashiwara-Vergne conjecture*, Lett. Math. Phys. 69 (2004), 177-203; arXiv:math.QA/0104100.

[5] Burgunder, E., *Eulerian idempotent and Kashiwara-Vergne conjecture*, arXiv:math.QA/0612548 (2006).

[6] Cattaneo, A. and Torossian, C., *Quantification pour les paires symétriques et diagrammes de Kontsevich*, arXiv:math.RT/0609693 (2006).

[7] Duistermaat, J.J., *On the similarity between the Iwasawa projection and the diagonal part*, Mémoire n°15 Soc. Math. France (1984), 129-138.

[8] Helgason, S., *Fundamental solutions of invariant differential operators on symmetric spaces*, Amer. J. Maths. 86 (1964), 565-601.

[9] Helgason, S., *Groups and geometric analysis*, Academic Press 1984.

[10] Helgason, S., *Geometric analysis on symmetric spaces*, Math. Surveys and Monographs 39, Amer. Math. Soc. 1994.

[11] Kashiwara, M. and Vergne, M., *The Campbell-Hausdorff formula and invariant hyperfunctions*, Invent. Math. 47 (1978), 249-272.

[12] Kontsevich, M., *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. 66 (2003), 157-216; arXiv:math.QA/9709040.

[13] Koornwinder, T., *A new proof of a Paley-Wiener type theorem for the Jacobi transform*, Arkiv för Mat. 13 (1975), 145-159.

[14] Rouvière, F., *Démonstration de la conjecture de Kashiwara-Vergne pour l'algèbre $sl(2)$*, C.R. Acad. Sci. Paris, Série I 292 (1981), 657-660.

[15] Rouvière, F., *Espaces symétriques et méthode de Kashiwara-Vergne*, Ann. Sci. École Norm. Sup. 19 (1986), 553-581.

[16] Rouvière, F., *Invariant analysis and contractions of symmetric spaces*, I, Compositio Math. 73 (1990), 241-270; II, Compositio Math. 80 (1991), 111-136.

[17] Rouvière, F., *Fibrés en droites sur un espace symétrique et analyse invariante*, J. Funct. Anal. 124 (1994), 263-291.

[18] Torossian, C., *Sur la conjecture combinatoire de Kashiwara-Vergne*, J. Lie Theory 12 (2002), 597-616.

[19] Torossian, C., *La conjecture de Kashiwara-Vergne [d'après Alekseev et Meinrenken]*, Séminaire Bourbaki n° 980 (Juin 2007).

[20] Vergne, M., *Le centre de l'algèbre enveloppante et la formule de Campbell-Hausdorff*, C.R. Acad. Sci. Paris, Série I 329 (1999), 767-772.

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