The Spherical Radon Transform and Support Theorems

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1 Outline

1. Helgason's proof of the support theorem for the classical hyperplane Radon transform and reduction to a theorem about the spherical transform.
2. Helgason's 1965 problem.
3. His answer and my answer.

In this talk, I'll start with Helgason's proof of the support theorem for the classical hyperplane Radon transform. He makes a lovely reduction to a support theorem for the spherical transform. Then, I'll give a conjecture of his for the spherical transform and both his and my answer.

I had hoped to talk about more of his theorems on the spherical transform and research that stemmed from them: Ásgeirsson's Mean value theorem, generalized Pizetti formulas, Huygens' principle, and his theorems about the geodesic transforms on projective spaces, but there was just too much good mathematics for me to do justice to it in 40 min.

As I read his articles, I was reminded what gems they are. Each is crystal clear with all details, and he writes valuable overview articles that make his research even more accessible. The results are fundamental and beautiful.

Personally, Sig has been a real mentor to me over the years. He was my first professor at MIT, and he taught me real analysis. He was so inspiring that we voted him math prof. of the year. He has always been generous of his time and ideas, when he was my co-Ph.D.-advisor, as I was getting started in the field, and as I prove new theorems now.

His geometric analysis seminars are always fun to attend. They are relaxed and we basically stand around and talk about ideas. He will even present proofs of other folks that he finds interesting.

Sig is very kind and thoughtful, and I am lucky to number him among my friends.

2 Support theorems in general

Radon transforms are integral transforms integrating over submanifolds.

Let M be a manifold and Ξ a collection of submanifolds of M.

For $f \in C_c(M)$ and $\xi \in \Xi$, define

The Radon transform: $Rf(\xi) = \displaystyle\int_{y\in\xi} f(y)dy_\xi$.

A support theorem looks like

Theorem $\mathcal{A} \subset \Xi$, ***[Hypotheses on*** f***,*** $\mathcal{A}$***.]*** *If* $Rf(\xi) = 0$ *for* $\xi \in \mathcal{A}$*, then* $\bigl(\operatorname{supp} f\bigr) \cap \bigl(\bigcup_{\xi\in\mathcal{A}} \xi\bigr) = \emptyset$.

Uses:

- Tomography
- Range theorems for C_c^∞ [He 1965].
- Solvability of invariant differential operators [He 1973].
- Converse of Huygens' Principle [He 1987].
- ....

Now study specific support theorems.

3 Radon transforms on $\mathbb{R}^n$

Classical spherical Radon transform:

For $x \in \mathbb{R}^n$ $r > 0$,

Sphere: $S(x,r) = \{y \in \mathbb{R}^n \,\big|\, |x-y| = r\}$

Disk: $D(x,r) = \{y \in \mathbb{R}^n \,\big|\, |x-y| \leq r\}$

$$SMf(x,r) = \int_{z \in S(x,r)} f(z) da(z)$$

(da normalized area measure)

Classical hyperplane Radon transform:

$\Xi =$ the set of $n-1$-dimensional hyperplanes in $\mathbb{R}^n$.

$f \in C_c(\mathbb{R}^n), \quad \xi \in \Xi \qquad\qquad Rf(\xi) = \displaystyle\int_{x\in\xi} f(x)\,dx_H$

Theorem 3.1. [Helgason, *Acta Math.*, **112**(1965), 153-180] *Let $f \in C(\mathbb{R}^n)$ satisfy the following conditions:*

i. For each integer $k > 0$, $|x|^k f(x)$ is bounded

ii. There is a constant $A > 0$ such that $Rf(\xi) = 0$ for ξ outside the disk $D(0, A)$.

Then $f(x) = 0$ outside of the disk $D(0, A)$.

The theorem also holds for hyperplanes outside of a convex set.

The proof is beautiful and it involves theorems on the spherical transform, so....

Proof. **I.** Reduce from $f \in C(\mathbb{R}^n)$ to $f \in C^\infty(\mathbb{R}^n)$: convolve with $\phi \in C_c^\infty(\mathbb{R}^n)$ supported in $D(0,\epsilon)$: $R(f * \phi) = (Rf) *_p (R\phi)$.

II. Prove the theorem for smooth quickly decreasing radial functions:

If $f(x) = F(|x|)$ radial, then $Rf(\xi) = \hat{F}(p)$ where $p = d(0,\xi)$.

$$\hat{F}(p) = \mathrm{Vol}(S^{n-1}) \int_p^\infty F(u)u(u^2 - p^2)^{(n-3)/2} du$$

This is a standard generalized Abel or Volterra integral equation and can be solved using tricks. The inversion formula involves a similar integral equation integrating from p to ∞. So,

If $\hat{F}(p) = 0$ *for* $p > A$*, then* $F(r) = 0$ *for* $r > A$. □

III. Reduce to a support theorem for SMf on spheres containing $D(0,A)$.

Let $x \in \mathbb{R}^n$ and $r > 0$. Now let

$$g_x(y) = \int_{k \in O(n)} f(x + ky)dk = SMf(x, |y|)\,.$$

Then g is radial in y and $R(g_x)(\xi) = \int_{k \in O(n)} Rf(x + k\xi)dk\,.$

Since $Rf(x+k\xi)=0$, if $d(0,\xi)>|x|+A$, the radial function g_x satisfies $Rg_x=0$ for $d(0,\xi)>|x|+A$.

Thus by **II.**, $g_x(y)=0$ for $|y|>|x|+A$.....$SMf(x,r)=0$ if $S(x,r)$ encloses $D(0,A)$ (i.e., $r>|x|+A$).

Theorem 3.2. *Assume* $f\in C^\infty(\mathbb{R}^n)$ *and* $|x|^k f(x)$ *is bounded for all* $k\in\mathbb{N}$. *Assume* $SMf(x,r)=0$ *for all spheres* $S(x,r)$ *enclosing* $D(0,A)$. *Then,* $f=0$ *outside of* $D(0,A)$.

Proof. Sig's idea is to perturb the center and show that integrals of (f times polynomials) over spheres containing $D(0,A)$ are zero.

Let $x\in\mathbb{R}^n$, $r>A+|x|$ then we know

$$\text{constant}=\int_{\mathbb{R}^n} f=\int_{D(x,r)} f=\int_{y\in D(0,r)} f(x+y)$$

If we take derivative with respect to the i^{th} coordinate we get

$$0 = \int_{D(0,r)} \frac{\partial}{\partial x_i} f(x+y)dy = \int_{D(0,r)} \frac{\partial}{\partial y_i} f(x+y)dy$$

Defining vector field $F(y) = f(x+y)\frac{\partial}{\partial y_i}$ and using the Divergence Theorem, we see

$$0 = \int_{D(0,r)} \operatorname{Div} F dy = \int_{S(0,r)} \langle F, \vec{n} \rangle = \int_{S(0,r)} f(x+y)\frac{y_i}{r}$$

Since f is quickly decreasing, we can continue and show for any polynomial $P(y)$ that

$$0 = \int_{S(0,r)} f(x+y)P(y)$$

and this shows $f(x+y) = 0$ for $y \in S(0,r)$ ($f = 0$ on $S(x,r)$).

$\square$

4 Helgason's problem and generalization

Soon after Sig proved Theorem 3.2 he asked [Problems in Differential Geometry, Kobayashi and Eells, Proc. US-Japan Seminar in Differential Geometry 1965]:

Problem 4.1. *Let M be a complete simply connected Riemannian manifold of negative curvature and D a closed ball in M. Let $f \in C_c^\infty(M)$. Assume f has surface integral 0 over every sphere enclosing D. Is $f = 0$ on $M \setminus D$?*

Sig answered this problem affirmatively for M of constant negative curvature (hyperbolic space) [He 1980].

Now my answer....

5 The sphere transform on manifolds.

Let M be a real-analytic Riemannian manifold and let $d(\cdot,\cdot)$ be the geodesic distance on M. For $y \in M$, $r > 0$

$$S(y,r) = \{z \in M \,\big|\, d(z,y) = r\}$$

$$D(y,r) = \{z \in M \,\big|\, d(z,y) \leq r\}$$

If $y \in M$, then we let I_y be the *injectivity radius of M at y*. $I_M = \inf\{I_y \,|\, y \in M\}$.

MORAL: If $r < I_y$ then $S(y,r)$ is diffeomorphic to a Euclidean sphere.

The set of spheres over which we integrate:

$$\Xi = \{(y,r) \in M \times (0,\infty) \,\big|\, \forall z \in D(y,r), r < I_z)\}$$

We include the injectivity radius assumption so that the Radon transform and its dual are well defined.

The *incidence relation* of points and spheres is

$$Z = \{(x,y,r) \,\big|\, (y,r) \in \Xi,\ x \in S(y,r)\}\,.$$

The *spherical Radon transform* for $f \in C(M)$ is

$$R_\mu f(y,r) = \int_{x \in S(y,r)} f(x)\mu(x,y,r)\, dM_r(x)\,.$$

We include the nowhere zero real-analytic weight $\mu(x,y,r)$ on Z since measures aren't canonical in general (and our proofs don't use symmetry).

Example: The spherical transform on all spheres in M. Injectivity??

MORAL: Injectivity is trivial unless you restrict the set of spheres.....look for support theorems, e.g., Helgason's problem.

Theorem 5.1 ([Grinberg & Q., *J. Funct. Anal.* 2000])**.** *Let M be a connected real-analytic Riemannian manifold, and let $\mathcal{A} \subset \Xi$ be connected and open. Let μ be nowhere-zero real-analytic weight on Z. Let $f \in \mathcal{D}'(M)$ and assume the* starter condition:

$$\exists (y_0, r_0) \in \mathcal{A} \text{ such that } S(y_0, r_0) \cap \operatorname{supp} f = \emptyset.$$

Assume $R_\mu f(y, r) = 0$ for all $(y, r) \in \mathcal{A}$. Then f is zero on the union of spheres, $\bigcup_{(y,r)\in\mathcal{A}} S(y, r)$.

For Helgason's theorem, $\mathcal{A}$ is the spheres enclosing the disk D.

Microlocal integral geometry: Guillemin, Sternberg, Agranovsky, Boman, Cheney, Finch, Greenleaf, Katsevich, Kuchment, Nolan, Phong, Stein, Stefanov, Uhlmann, ETC.

Using Radon transforms to understand microolocal analsysis: Guillemin, Sternberg, Candes, Kaneko, Tak., ETC.

Some of my support theorems:

Spheres of one radius in real-analytic manifold: [Q 1993]

Spheres with centers restricted to a real-analytic hypersurface in $\mathbb{R}^2$: [AQ 1996], $\mathbb{R}^n$: [AQ 2000, 2003, 2006],
real analytic manifolds: [Q2006]

Relation to stationary sets for wave equation: [AQ 1996, 2000, 2003, 2006]

6 Proof outline of Theorem 5.1

Big ideas

1. Use a good concept of singularity: exponential decrease of localized Fourier transform at the point $x \in M$, $\eta \in T_x^*(M)$: $(x, \eta) \in \mathrm{WF}_A(f)$.
2. Tell what R_μ does to $\mathrm{WF}_A(f)$ (microlocal regularity).
3. Give the behavior of $\mathrm{WF}_A(f)$ near $\mathrm{bd}(\mathrm{supp}\, f)$ (KKH).
4. Let spheres $S(y, r)$ for $(y, r) \in \mathcal{A}$ eat away at $\mathrm{supp}\, f$.

Definition 6.1. Let $f \in \mathcal{E}'(\mathbb{R}^n)$ and let $x_1 \in \mathbb{R}^n$ and $\eta_1 \in T^*_{x_1}(\mathbb{R}^n)\setminus 0$. Then, $(x_1, \eta_1) \notin \mathrm{WF}_A(f)$ **iff** there are neighborhoods U of x_1 and V of η_1 and $\exists C > 0$, $\exists c > 0$ such that for all $x \in U$, $\eta \in V$: $\left| \displaystyle\int_{y \in \mathbb{R}^n} \left(f(y) e^{-\lambda|x-y|^2}\right) e^{-iy\cdot(\lambda\eta)} dy \right| \le C e^{-c\lambda}$.

(Extends to $f \in \mathcal{D}'(\mathbb{R}^n)$ by localizing and to manifolds using coordinates.)

The Radon transform detects wavefront set in a precise way.

Theorem 6.2 (Microlocal Regularity Theorem)**.** *Let $f \in \mathcal{D}'(M)$. Let μ be a nowhere zero real-analytic weight on Z and let R_μ be the associated Radon transform on spheres in Ξ. Let $S_1 = S(y_1, r_1) \in \Xi$. Let $x_1 \in S_1$ and let η_1 be conormal to S_1 at x_1. If $R_\mu f$ is zero in a neighborhood of (y_1, r_1), then $(x_1, \eta_1) \notin \mathrm{WF}_A(f)$.*

Reason: R_μ is an elliptic Fourier integral operator (assoc. to $N^*Z\setminus 0$) and so R_μ detects only singularities conormal to spheres of integration.

R_μ satisfies the microlocal Bolker assumption so $R_\mu^* R_\mu$ is an analytic elliptic pseudodifferential operator (f, Z localized).

Proof outline of Theorem 6.2:

- Use the definition of R_μ as a composition of a push-forward multiplication by μ and pull-back to localize at x_1 [SKK, Wakabayashi, Hö].

- Use either stationary phase [Stefanov-Uhlmann] or asymptotic arguments [Boman] to show that $R_\mu^* R_\mu$ is a microlocally elliptic operator near (x_1, η_1). [Leiss, Rodino] (Near x_1, the spheres are essentially hyperplanes, and they are conormal all η sufficiently near η_1.) $\square$

An important theorem of Kawai, Kashiwara and Hörmander gives precise information about analytic wavefront set at $\operatorname{bd}(\operatorname{supp} f)$.

Theorem 6.3 (Kawai, Kashiwara, and Hörmander)**.** *Let* $f \in \mathcal{D}'(M)$. *Let* $x_1 \in \operatorname{bd}(\operatorname{supp} f)$ *and assume the sphere* $S_1 = S(y_1, r_1)$ *has* $x_1 \in S_1$; S_1 *is to one side of* $\operatorname{bd}(\operatorname{supp} f)$ *near* x_1; *and* η_1 *is conormal to* S_1 *at* x_1.

Then, $(x_1, \eta_1) \in \mathrm{WF}_A(f)$.

(f cannot be real-analytic at x_1)

Now we draw a contradiction.

Take a path in $\mathcal{A}$ from $S(y_0, r_0)$ to a sphere that meets $\operatorname{supp} f$.

Let $S(y_1, r_1)$ be the first sphere on the path that meets $\operatorname{supp} f$ and let $x_1 \in S(y_1, r_1) \cap \operatorname{supp} f$. Let η_1 be conormal to $S(y_1, r_1)$ at x_1.

KKH $\Longrightarrow (x_1, \eta_1) \in \mathrm{WF}_A(f)$.

Microlocal regularity (Theorem 6.2) $\Longrightarrow (x_1, \eta_1) \notin \mathrm{WF}_A(f)$.

7 Observations

- KKH fails in the C^∞ category...we need analyticity.
- Our proof doesn't use symmetry like Sig's proof so it's valid for more general manifolds and measures, *but* Sig's theorem is valid for functions not of compact support (and without a starter condition).
- Sig's perturbation argument is morally related to microlocal regularity: We need to be able to perturb centers and radii enough so we can find spheres near $S(y_1, r_1)$ conormal to all covectors in a neighborhood of (x_1, η_1).