Holomorphic extension
the Radon transform
the heat equation
and
other problems in harmonic analysis

Joint with H. Schlichtkrull
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- where h_t is the heat kernel on M, ie., $h_t \geq 0$, $h_t(x,\cdot)d\sigma$ a probability measure on M and $h_t(x,y) = h_t(y,x)$.

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- $u(gx_o, t) = \int_{G/K} f(hx_o)h_t(h^{-1}g)\,dh_K$.

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- In both cases weighted only in the $y \in T_x^*(\mathbb{R}^n)$ direction.

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- $\hat{f}_r(\omega) = \hat{f}(r\omega)$
- The natural way to realize those reps is by using the Radon transform

$$\mathcal{R}f(w, r) = \int_{x\cdot w = r} f(x)\, dx$$

- G-intertwining $C_c(\mathbb{R}^n) \to C_{\text{even}}(\Xi)$, $\Xi = S^{n-1} \times \mathbb{R} = G/MN$, $M = \mathrm{SO}(n-1)$, $N = \mathbb{R}^{n-1}$ and

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- **Theorem** *The operator Λ extends to an unitary intertwining operator*

$$\Lambda : (L^2(X), L_X) \to (L^2(\Xi), L_\Xi) \simeq \int_{\mathbb{R}^+}^{\oplus} (L^2(S^{n-1}), \pi_r)\, dr\,.$$

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- $\Rightarrow p \mapsto H_t^{\mathbb{R}}(\Lambda f)(\cdot,p)$ extends to a holomorphic $L^2(S^{n-1})$-valued function on $\mathbb{C}$. For $F \in \mathcal{F}_t(\mathbb{C}^n)$ take $f \in L^2(X)$ such that $H_t(f) = F$ and define $\widetilde{\Lambda}(F)$ to be the holomorphic extension of $H_t^{\mathbb{R}}(\Lambda f)$.

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- **Theorem** *The map* $\widetilde{\Lambda} : \mathcal{F}_t(\mathbb{C}^n) \to \mathcal{F}_{\mathbb{Z}_2,t}(S^{n-1} \times \mathbb{C})$ *is an unitary* G*-isomorphism.*

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- B. Hall and J.J. Mitchell, 2004: The case $\mathcal{M} = G/K$ where G is complex or of rank one

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- The general case in 2007.

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- $e_{\lambda,b}(x) = a(x^{-1}k)^{-\lambda-\rho}$ Fourier tranf.

$$\hat{f}_\lambda(b) = \pi_{-\lambda}(f) p_{-\lambda}(b) = \int_M f(x) e_{-\lambda,b}(x) \, dx$$

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- *Finally* $\widehat{(Lf)}(\lambda, b) = -(\lambda^2 + |\rho|^2)\widehat{f}(\lambda, b)$.

The convexity theorem

- Let $\Omega := \{X \in \mathfrak{a} \mid (\forall \alpha \in \Sigma)\, |\alpha(X)| < \pi/2\}$ and $\mathrm{Cr}(X) := G \exp i\Omega \cdot x_o$ the crown, an open G-invariant domain in $\mathcal{M}_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$, called the complex crown.

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- Let $\omega(i\nu) = \sup_{Y\in\Omega} \frac{1}{|W|} \sum_{w\in W} e^{2\nu(w(Y))}$. Then $Y \in \Omega \Rightarrow \exists T \in \mathfrak{a}^+$, $g \in G\ \exists C > 0$, such that

$$|e_{i\nu,b}(g \exp iY \cdot x_o)| \leq C\sqrt{\omega(i\nu)}e^{-\nu(T)}\,.$$

for all $\nu \in \mathfrak{a}^*_+$.

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- All the spherical functions and the heat kernel extends to holomorphic functions on $\mathrm{Cr}(X)$ (Krötz+Stanton).
- **Theorem (Gindikin-Krötz-Otto)** *Let $Y \in \Omega$. Then*

$$a(\exp(iY)G) = A \exp(i\mathrm{conv}(W \cdot Y))\,.$$

- Let $\omega(i\nu) = \sup_{Y\in\Omega} \frac{1}{|W|} \sum_{w\in W} e^{2\nu(w(Y))}$. Then $Y \in \Omega \Rightarrow \exists T \in \mathfrak{a}^+$, $g \in G$ $\exists C > 0$, such that

$$|e_{i\nu,b}(g \exp iY \cdot x_o)| \leq C\sqrt{\omega(i\nu)}e^{-\nu(T)}\,.$$

for all $\nu \in \mathfrak{a}^*_+$.

The Hardy space

- $\mathcal{H}_X$: $F \in \mathcal{O}(\mathrm{Cr}(X))$ such that $F|_X \in L^2(X)$ and

$$\|F\|^2_{\mathcal{H}_X} := \frac{1}{|W|} \int_{i\mathfrak{a}^* \times B} |\widehat{F|_X}(\lambda, b)|^2 \, \underbrace{\omega(\lambda) d\mu(\lambda, b)}_{d\mu_\omega} < \infty \, .$$

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- **Theorem (Ó-S)** The space $\mathcal{H}_X$ is a G-invariant Hilbert space. The action of G is unitary and

$$(\mathcal{H}_X, L)) \simeq \int_{i\mathfrak{a}^*_+}^{\oplus} (H_{-\lambda}, \pi_{-\lambda}) \, d\mu_\omega(\lambda) \, .$$

Furthermore, the following holds:

1. Let $F \in \mathcal{H}_X$ and $f = F|_X$. Then for all $z \in \mathrm{Cr}(X)$:

$$F(z) = \int \widehat{f}(\lambda, b) e_{\lambda,b}(z)\, d\mu(\lambda, b)\,.$$

2. For each $\varphi \in L^2(i\mathfrak{a}_+^* \times B, \omega d\mu)$ the function defined by

$$F(z) = \int \varphi(\lambda, b) e_{\lambda,b}(z)\, d\mu(\lambda, b)$$

   belongs to $\mathcal{H}_X$ and has $\widehat{F|_X} = \varphi$.

3. $\mathcal{H}_X$ is a reproducing kernel Hilbert space with reproducing kernel

$$\begin{aligned} K(z,w) &= \int_{i\mathfrak{a}_+^* \times B} \frac{e_{\lambda,b}(z) e_{-\lambda,b}(\sigma(w))}{\omega(\lambda)}\, d\mu(\lambda, b) \\ &= \int_{i\mathfrak{a}_+^*} \frac{\varphi_\lambda(\sigma(w)^{-1} z)}{\omega(\lambda)}\, d\mu(\lambda) \end{aligned}$$

The Segal Bargmann transform

- Let $u(x,t) = H_t f(x)$. Then

$$u(x,t) = \frac{1}{|W|} \int_{i\mathfrak{a}^* \times B} e^{-t(|\lambda|^2 + |\rho|^2)} \hat{f}(\lambda, b) e_{\lambda,b}(x) \, d\mu(\lambda, b) \, .$$

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- Hence $x \mapsto u(x,t)$ extends to a function in $\mathcal{H}_X$.
- $L^2(\mathcal{M}) \ni f \mapsto H_t f \in \mathcal{O}(\mathrm{Cr}(X))$ is the Segal-Bargmann transform.

The Radon transform

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- It intertwines the regular action on $C_c^\infty(\mathcal{M})$ and the action on $C(B \times A)$:

$$\tau(g)\varphi(b, a) = a(g^{-1}b)^{-\rho}\varphi(k(g^{-1}b), a(g^{-1}ba)) \,.$$

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- $L^2_W(B \times i\mathfrak{a}^*, |W|^{-1} db\, d\lambda) \simeq L^2_W(i\mathfrak{a}^*, L^2(B); |W|^{-1}\, d\lambda)$ the space of functions in $L^2(B \times i\mathfrak{a}^*, |W|^{-1} db\, d\lambda)$ such that

$$(\forall w \in W) \quad c(-w\lambda)F(\cdot, w\lambda) = c(-\lambda)\mathcal{A}(w, -\lambda)F(\cdot, \lambda)\,,$$

and

$$L^2_W(B \times A) := \mathcal{F}_A^{-1}(L^2_W(B \times i\mathfrak{a}^*, |W|^{-1} db\, d\lambda))\,.$$

Unitary G-isomorphism

- **Theorem** $\Lambda = (\mathrm{id} \times \Psi) \circ \mathcal{R}_\rho : L^2(\mathcal{M}) \to L^2_W(B \times A)$ *is an unitary* G*-isomorphism. Furthermore* $\Lambda(L_{\mathcal{M}} f) = (L_A - |\rho|^2)\Lambda(f)$.

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- From this it follows that:
  Lemma *Let $f \in L^2(X)$. Then $e^{t|\rho|^2}\Lambda(H_t f)$ solves the heat equation on A with initial value $\Lambda(f) \in L^2(B)$. In particular, the map $\mathfrak{a} \ni X \mapsto \Lambda(H_t f)(\cdot, \exp X) \in L^2(B)$ extends to a unique holomorphic function on $\mathfrak{a}_{\mathbb{C}}$, again denoted by $\Lambda(H_t f)$ such that*

$$\int_{B\times\mathfrak{a}_{\mathbb{C}}} |e^{t|\rho|^2}\Lambda(H_t f)(b, \exp(X+iY))|^2\, e^{-|Y|^2/2t}\, dbdXdY < \infty\,.$$

The Fock-space

- $\mathcal{F}_{W,t}(B \times \mathfrak{a}_{\mathbb{C}})$ the space of functions F on $B \times \mathfrak{a}_{\mathbb{C}}$ such that
$$\mathfrak{a}_{\mathbb{C}} \ni Z \mapsto F(\cdot, Z) \in L^2(B)$$
is holomorphic with $F \circ (\mathrm{id}, \log) \in L^2_W(B \times A)$, and satisfies
$$\|F\|_t^2 = \frac{1}{|W|(2\pi t)^{r/2}} \int_{\mathfrak{a}_{\mathbb{C}}} \|F(X+iY)\|^2_{L^2(B)}\, e^{-|Y|^2/2t}\, dX dY < \infty\,.$$

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- For $t>0$ define $\Lambda_t:\mathcal{O}_t(\mathrm{Cr}(X))\to\mathcal{F}_{W,t}(B\times\mathfrak{a}_{\mathbb{C}})$ in the following way. Let $F\in\mathcal{O}_t(\mathrm{Cr}(X))$. There exists a unique $f\in L^2(X)$ such that $F|_X=H_tf$. Let $\Lambda_t(F)$ be the holomorphic extension of $e^{t|\rho|^2}\Lambda(H_tf)$. Then $\|\Lambda_t(F)\|_t<\infty$, and the Weyl group relations are satisfied.

The image of S-B

- **Theorem (Ó-S)** *The map* $\Lambda_t : \mathcal{O}_t(\mathrm{Cr}(X)) \longrightarrow \mathcal{F}_{W,t}(B \times \mathfrak{a}_{\mathbb{C}})$ *is an unitary isomorphism. Furthermore, let* $F \in \mathcal{O}_t(\mathrm{Cr}(X))$*. Define* $f \in L^2(X)$ *by applying* Λ^* *to the function on* $B \times A$ *given by*

$$(b, a) \mapsto (4\pi t)^{-r/2} \lim_{R\to\infty} \int_{|Y|\leq R} \Lambda_t(F)(b, \log a + iY) e^{-|Y|^2/4t} \, dY \, .$$

Then $H_t(f) = F$*.*

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  Then $H_t(f) = F$*.*
- Still open problem: Direct limits!

The Radon transform again

- A complex horocycle in $\mathcal{M}_{\mathbb{C}}$ is a set of the form $gN_{\mathbb{C}} \cdot x_o$, $g \in G_{\mathbb{C}}$. If $\xi_o^{\mathbb{C}} = N_{\mathbb{C}} \cdot x_o$. Then as a $G_{\mathbb{C}}$ set the set $\Xi_{\mathbb{C}}$ of complex corocycles is $G_{\mathbb{C}}/M_{\mathbb{C}}N_{\mathbb{C}}$.

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- $\|F\|^2_{\mathcal{H}_\Xi} := \sup_{Y \in \Omega} \int_{B \times A} |F(b, a \exp iY)|^2 \frac{dbda}{|W|} < \infty$.

Image of Λ

- Let $F \in \mathcal{H}_X$ and $\varphi = F|_X$. Let $t_n \to 0$, $t_n > 0$, and view $\varphi_n := H_{t_n}\varphi$ as an element of $\mathcal{H}_X$. Then $\lim_n \Lambda(H_{t_n}\varphi)$ exists in $\mathcal{H}_\Xi$ and is independent of the sequence t_n. Define $\tilde{\Lambda} : \mathcal{H}_X \to \mathcal{H}_\Xi$ by

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- **Theorem (Ó+S)** *The map* $\tilde{\Lambda} : \mathcal{H}_X \to \mathcal{H}_\Xi$ *is an unitary intertwining isomorphism.*