Branchings to symmetric pairs and analysis on symmetric spaces

International Conference on Integral Geometry, Harmonic Analysis and Representation Theory

in honor of Sigurdur Helgason on the occasion of his 80th birthday

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Questions in Representation Theory

Analysis and Synthesis

- Find smallest objects
- Decompose into / Build up from } smallest objects

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  - Classification of irreducible reps
  - Decomposition into irreducible reps
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    - Induction
    - Restriction
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- Outside
  - Modular varieties, discont gps
  - Geometric analysis
  - Algebraic analysis
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Decomposition into irreducible reps

Two important cases

$G' \underset{\text{subgroup}}{\subset} G$

1) Induction

2) Restriction

Decomposition into irreducible reps

Two important cases

$G' \underset{\text{subgroup}}{\subset} G$

1) Induction: $G' \uparrow G$

Plancherel Formula

(e.g. Analysis on homo. space G/G')

2) Restriction: $G \downarrow G'$

Branching Law

(e.g. Tensor product, . . .)

Notation

Throughout this talk,

Ex.

G/K: Riemannian symmetric sp. $GL(n, \mathbb{R})/O(n)$

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$\Downarrow$ more general

G/H: reductive symmetric sp. $\quad GL(n,\mathbb{R})/O(p,n-p)$

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$\Downarrow$ more general

G/H: reductive symmetric sp. $\quad GL(n,\mathbb{R})/O(p,n-p)$

i.e. G reductive Lie gp

$\tau \in \mathrm{Aut}(G),\ \tau^2 = \mathrm{id}$

$H =$ open subgp of G^τ

$H \subset G$ reductive symmetric pairs

$$
\begin{array}{c}
\vdots \\
\cap \\
O(p,q) \\
\cap \\
U(p,q) \\
\cap \\
Sp(p,q) \\
\cap \\
U(2p,2q) \\
\cap \\
O(4p,4q) \\
\cap \\
\vdots
\end{array}
$$

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\begin{array}{cc}
\vdots & \vdots \\
\cap & \cap \\
O(p,q) & GL(n,\mathbb{R}) \\
\cap & \cap \\
U(p,q) & Sp(n,\mathbb{R}) \\
\cap & \cap \\
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\cap & \cap \\
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$n = p + q$

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\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \cap & & \cap & & \cap & & \\
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 & & \cap & & \cap & & \cap & & \\
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 & & \cap & & \cap & & \cap & & \\
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(G, K) **v.s.** (G, H)

G/K

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$$G/K$$

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- Restriction from G to K
  Harish-Chandra, Lepowski, Hua–Kostant–Schmid, Blattner Conjecture (Hecht–Schmid),Vogan, Bernstein, . . .

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Any unitary rep π can be decomposed into a direct integral of irreducible unitary reps:

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$$n(\tau) \in \mathbb{N} \cup \{\infty\}$$

Multiplicities

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- $L^2(G/K)$ 0 or 1 (Cartan '29, Gelfand '50)
- $G \downarrow K$ finite (Harish-Chandra's admissibility thm)
- $L^2(G/H)$ uniformly bounded (van den Ban)

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$L^2(G/H)$	uniformly bounded	(van den Ban)
$G \downarrow H$	can be ∞ but . . .	

- $G \downarrow H$: $\underbrace{\text{can be } \infty}_{\text{(usually) bad feature}}$ $\underbrace{\text{but} \ldots}_{\text{(unexpectedly) nice feature}}$

65th birthday conference (1992)

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G/H	$G \downarrow H$

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Exist $\pi \in \widehat{G}$ and (G, H) s.t. $\pi|_H$ is (almost) irreducible

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	(G, H)	$\pi \in \widehat{G}$
e.g.	$(U(2p, 2q),\ Sp(p, q))$	most degen. $A_{\mathfrak{q}}(\lambda)$
e.g.	$(GL(2n, \mathbb{R}),\ Sp(n, \mathbb{R}))$	most degen. p.s.
e.g.	$(O(p, q),\ O(p-1, q))$ $(p+q$: even$)$	minimal rep.

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G/K	$G \downarrow K$
G/H	$G \downarrow H$

$$\widehat{G} \ni \pi \leftarrow \cdots\cdots \rightarrow \mathcal{O}_\pi \in \mathfrak{g}^* / \operatorname{Ad}^*(G)$$

orbit method

Observation $G \downarrow H$ can be simpler than $G \downarrow K$

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This happens when $^\exists$open H-orbit on $\mathcal{O}_\pi$

(Unexpectedly) nice features

$$\boxed{\begin{matrix} G/K & G \downarrow K \\ G/H & G \downarrow H \end{matrix}} \qquad \pi \in \widehat{G}$$

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$$\begin{array}{ll} G/K & G \downarrow K \\ G/H & G \downarrow H \end{array} \qquad \pi \in \widehat{G}$$

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Observation $G \downarrow H$ can be simpler than $G \downarrow K$

- $\pi|_H$ may be (almost) irreducible, whereas $\pi|_K$ always decomposes into ∞ many K-types.

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G/H	$G \downarrow H$	$\pi \in \widehat{G}$

Observation $G \downarrow H$ can be simpler than $G \downarrow K$

- $\pi|_H$ may be (almost) irreducible, whereas $\pi|_K$ always decomposes into ∞ many K-types.

  $\Longrightarrow$ application to analysis on certain non-symmetric homo. sp.

  (e.g. $G_2(\mathbb{R})/SL(3,\mathbb{R})$, $G_2(\mathbb{R})/SU(2,1)$, . . .)

  $\cdots\cdots$ Helgason 65th birthday conference (1992)

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G/H	$G \downarrow H$	$\pi \in \widehat{G}$

<u>Observation</u> $G \downarrow H$ can be simpler than $G \downarrow K$

- Branching law for $\pi|_H$ may be computable, even when K-type formula (branching law for $\pi|_K$) is complicated.

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- Branching law for $\pi|_H$ may be computable, even when K-type formula (branching law for $\pi|_K$) is complicated.

  $\Longrightarrow$ application to certain $\pi = A_{\mathfrak{q}}(\lambda)$ to determine explicitly $\{\lambda : A_{\mathfrak{q}}(\lambda) \neq 0\}$

  (K– , Memoirs AMS '92)

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G/K	$G \downarrow K$
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Observation $G \downarrow H$ can be simpler than $G \downarrow K$

- $\pi|_H$ may be multiplicity-free, even when $\pi|_K$ is not multiplicity-free.

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  $\Longrightarrow$ application to certain degenerate principal series (Lee–Loke, Compostio Math '03; Matumoto)

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<u>Observation</u> $G \downarrow H$ can be simpler than $G \downarrow K$

- The decomposition of the restriction $\pi|_H$ may construct 'new' irreducible unitary representations of H as discrete summand
  (Howe, Gross–Wallach '94, Zhang '04)
  ((G, H) not necessarily symmetric)

Branching $G \downarrow H \Rightarrow$ Plancherel $L^2(G'/H')$

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Branching $G \downarrow H \Rightarrow$ **Plancherel** $L^2(G'/H')$

G'/K'	$\nwarrow$	$G \downarrow K$
G'/H'	$\longleftarrow$	$G \downarrow \not{H}\ G'$

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Special cases of restriction $\pi|_{G'}$ can be unitarily equivalent to $L^2(G'/H')$ (concretely/abstractly).

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- $G'/K' = GL(n, \mathbb{R})/O(n)$

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Special cases of restriction $\pi|_{G'}$ can be unitarily equivalent to $L^2(G'/H')$ (concretely/abstractly).

- $G'/K' = GL(n, \mathbb{R})/O(n)$
  $\Leftarrow (G, \pi) = (Sp(n, \mathbb{R}),$ holo. disc. series)
  (Ólafsson–Ørsted, . . .)
- $G'/H' = GL(n, \mathbb{R})/GL(p, \mathbb{R}) \times GL(n - p, \mathbb{R})$
  $\Leftarrow (G, \pi) = (G' \times G',$ certain degenerate principal series)
  ('canonical rep' of Gelfand–Graev–Vershik, van Dijk, Molchanov, . . .)

Learning from G/K

$$
\begin{array}{ccccc}
G/K & & G \downarrow K & & \\
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<u>Problem</u> Find a theory for $G \downarrow H$
as a counterpart to the following G/K result.

Learning from G/K

G/K		$G \downarrow K$	
	$\searrow$	$\vdots$	$\pi \in \widehat{G}$
G/H	$\cdots\cdots$	$G \downarrow H$	

Problem   Find a theory for $G \downarrow H$
as a counterpart to the following G/K result.

- $L^2(G/K)$ is multiplicity-free
- G compact: Cartan–Helgason formula
- G non-compact: no discrete series for G/K

Multiplicity-free theorem

G/K	$G \downarrow K$
G/H	$G \downarrow H$

Multiplicity-free theorem

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Fact $\quad L^2(G/K)$ is multiplicity-free.

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$$\begin{matrix} G/K & G \downarrow K \\ G/H & G \downarrow H \end{matrix}$$

Fact $L^2(G/K)$ is multiplicity-free.

$\updownarrow$

Theorem A (to appear in Progr Math)

$\pi_\lambda \otimes \pi_\mu, \pi_\lambda|_H$ are multiplicity-free.

Here, $\pi_\lambda \in \widehat{G}$ stands for

- (G: compact) gen. rectangular shaped representation
- (G: non-compact) highest wt rep of scalar type

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Examples of Thm A

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π_λ, π_μ: gen rectangular-shaped rep / highest wt rep of scalar type

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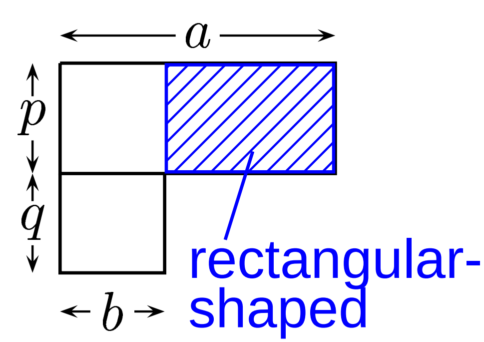
Ex. $G = U(n),\ n = p + q = i + j$

$\lambda = (a, \dots, a, b, \dots, b) \in \mathbb{Z}^{p+q}\ (a \geq b)$

$H = U(i) \times U(j),\ O(n)$



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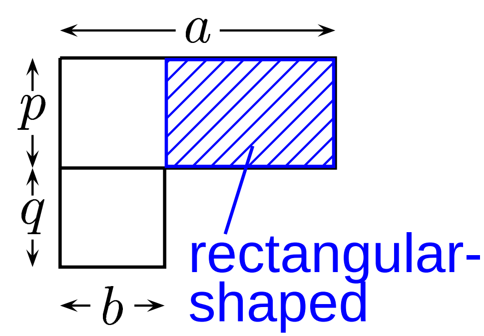
Thm A $\pi_\lambda \otimes \pi_\mu, \pi_\lambda|_H$ are multiplicity-free.

π_λ, π_μ: gen rectangular-shaped rep / highest wt rep of scalar type

Ex. $G = U(n),\ n = p + q = i + j$

$\lambda = (a, \ldots, a, b, \ldots, b) \in \mathbb{Z}^{p+q}\ (a \geq b)$

$H = U(i) \times U(j),\ O(n)$



Ex. $G = U(p, q)$

π_λ : highest wt rep of minimal K-type $(\det)^a \otimes (\det)^b$

$H = U(i, j) \times U(p - i, q - j),\ O(p, q),\ \ldots$

Gen. rectangular-shaped rep

G: compact Lie gp

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fundamental wt

Def. π_λ is a gen. rectangular representation

$\iff \lambda \in \mathbb{N} \cdot \omega_i$ where α_i occurs in the highest root with coeff 1

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<u>Def.</u> π_λ is a gen. rectangular representation

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$\iff \pi_\lambda$ is realized in holo sections for equiv holo line b'dle over compact Hermitian symm space of G

Unitary highest wt rep

G non-compact, simple Lie gp, G/K Hermitian

Ex. $SU(p,q), SO(n,2), Sp(n,\mathbb{R})$, $SO^*(2n), E_{6(-14)}, E_{7(-25)}$

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}^+ + \mathfrak{p}^-$$

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Def. $(\pi, V) \in \widehat{G}$ unitary highest wt rep

$\iff \{v \in V^\infty : d\pi(X)v = 0 \ ({}^\forall X \in \mathfrak{p}^+)\} \neq 0$

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μ K

Write $\pi = \pi^G(\mu)$ $(\mu \in \widehat{K})$ or simply π_μ

Def. π : holomorphic discrete series

$\iff \mathrm{Hom}_G(\pi, L^2(G)) \neq 0$

π : scalar type $\iff \dim \mu = 1$

Proof of Thm A (step 1)

Method: Propagation Theorem of MF property

MF $=$ multiplicity-free

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Theorem (Propagation Theorem, math.RT/0607004)

$G_x \curvearrowright \mathcal{V}_x$ MF $(\forall x \in D)$

$\Longrightarrow G \curvearrowright \mathcal{O}(D, \mathcal{V})$ MF

if the G-action on D is 'strongly visible'.

Idea goes back to Gelfand, Siegel, Faraut–Thomas, S.Kobayashi.

Propagation of MF property

Points G, D can be non-compact

- propagation thm of MF property

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$$
\begin{array}{ccc}
 & & \mathcal{V} \\
 & \curvearrowright & \\
G & & \downarrow \\
 & \curvearrowright & \\
 & & D
\end{array}
$$

fiber MF

Propagation of MF property

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fiber MF ⟹ sections MF

Propagation of MF property

Points G, D can be non-compact

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$$G \curvearrowright \begin{matrix} \mathcal{V} \\ \downarrow \\ D \end{matrix} \quad \Longrightarrow \quad G \curvearrowright \mathcal{O}(D, \mathcal{V})$$

fiber MF $\qquad$ sections MF

$$\Uparrow$$

geometry of base space
. . . '(strongly) visible action'

Visible actions

holomorphic

$G \curvearrowright (D, J)$ complex mfd, connected

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Def. Action is visible if

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$^{\exists} N \subset D'$ s.t. totally real

$$\begin{cases} N \text{ meets every } G\text{-orbit} \\ J_x(T_x N) \subset T_x(G \cdot x) \; (x \in N) \end{cases}$$

Example of (strongly) visible actions

$$\mathbb{T} = \{a \in \mathbb{C} : |a| = 1\} \quad (\simeq S^1)$$

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$\mathbin{\rotatebox[origin=c]{90}{$\in$}}$

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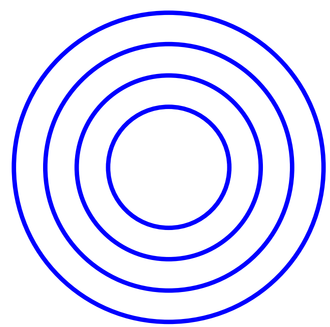

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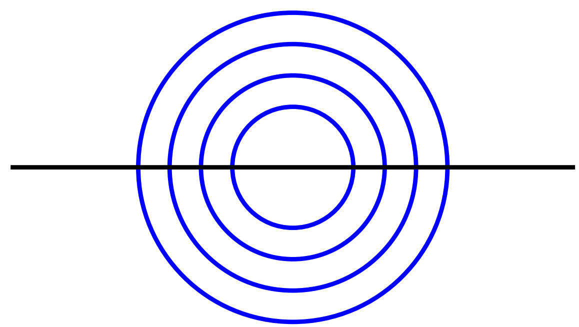

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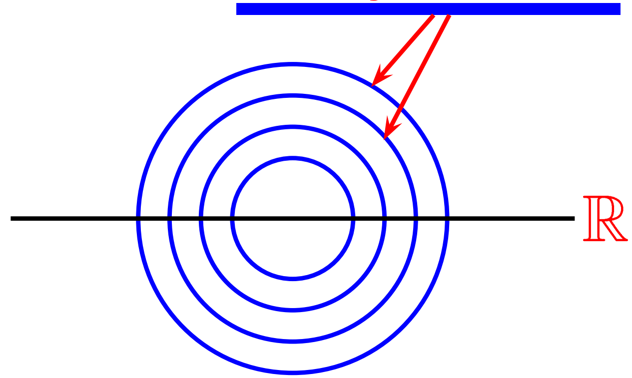

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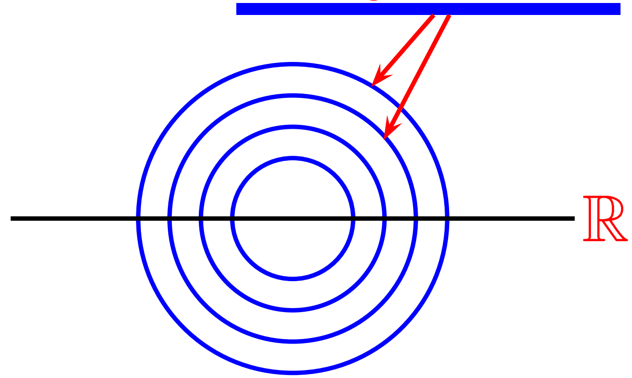

$\Longrightarrow$ $\mathbb{T}$-action on $\mathbb{C}$ is visible

Strongly visible actions

holomorphic
$G \curvearrowright D$ complex mfd, connected

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Def. The action is (strongly) visible

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σ: anti-holo; $D' \subset D$: open

s.t. $\begin{cases} \sigma|_N = \mathrm{id} \\ N \text{ meets every } G\text{-orbit in } D' \\ \sigma \text{ stabilizes every } G\text{-orbit.} \end{cases}$

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Proposition (K– , Publ RIMS '04)

Strongly visible $\Longrightarrow$ Visible

Visible / polar / coisotropic

Podestà–Thorbergsson (TAMS '02) & K– (Publ RIMS '05)

$$\underset{\text{compact}}{H} \curvearrowright \underset{\text{compact, Kähler}}{D}$$

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Riemannian

Polar

Visible

Complex

Coisotropic (multiplicity-free)

Symplectic

Visible / polar / coisotropic

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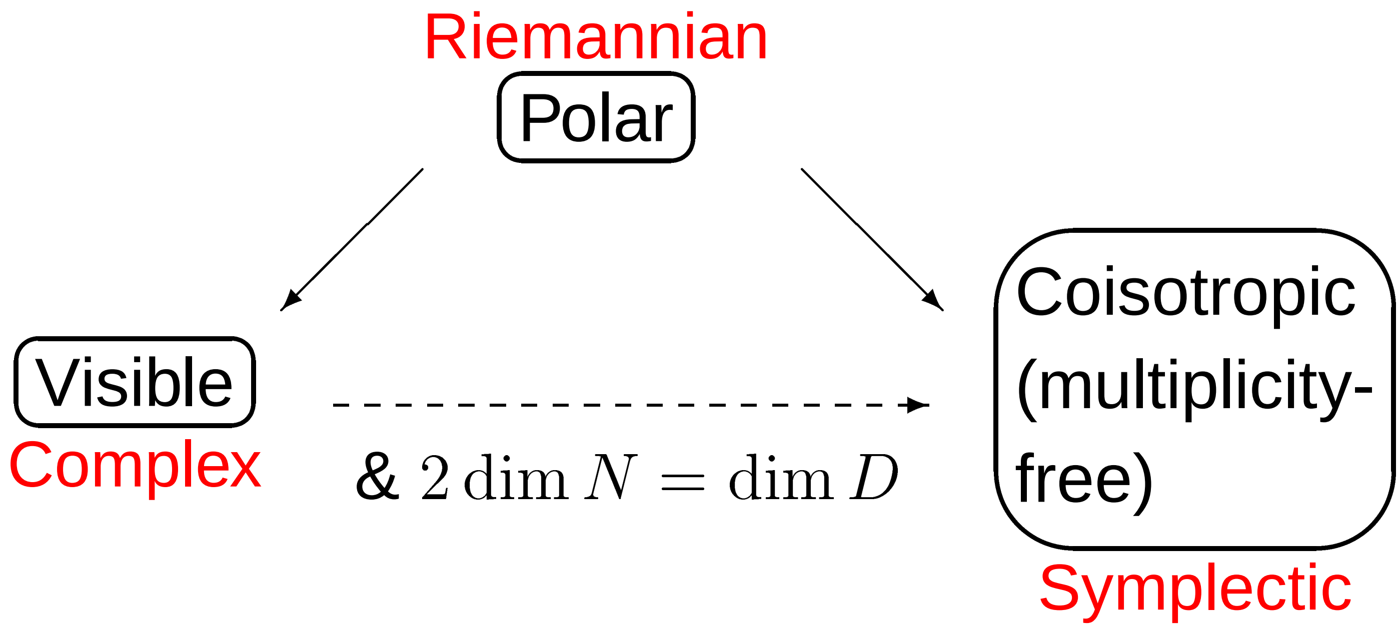

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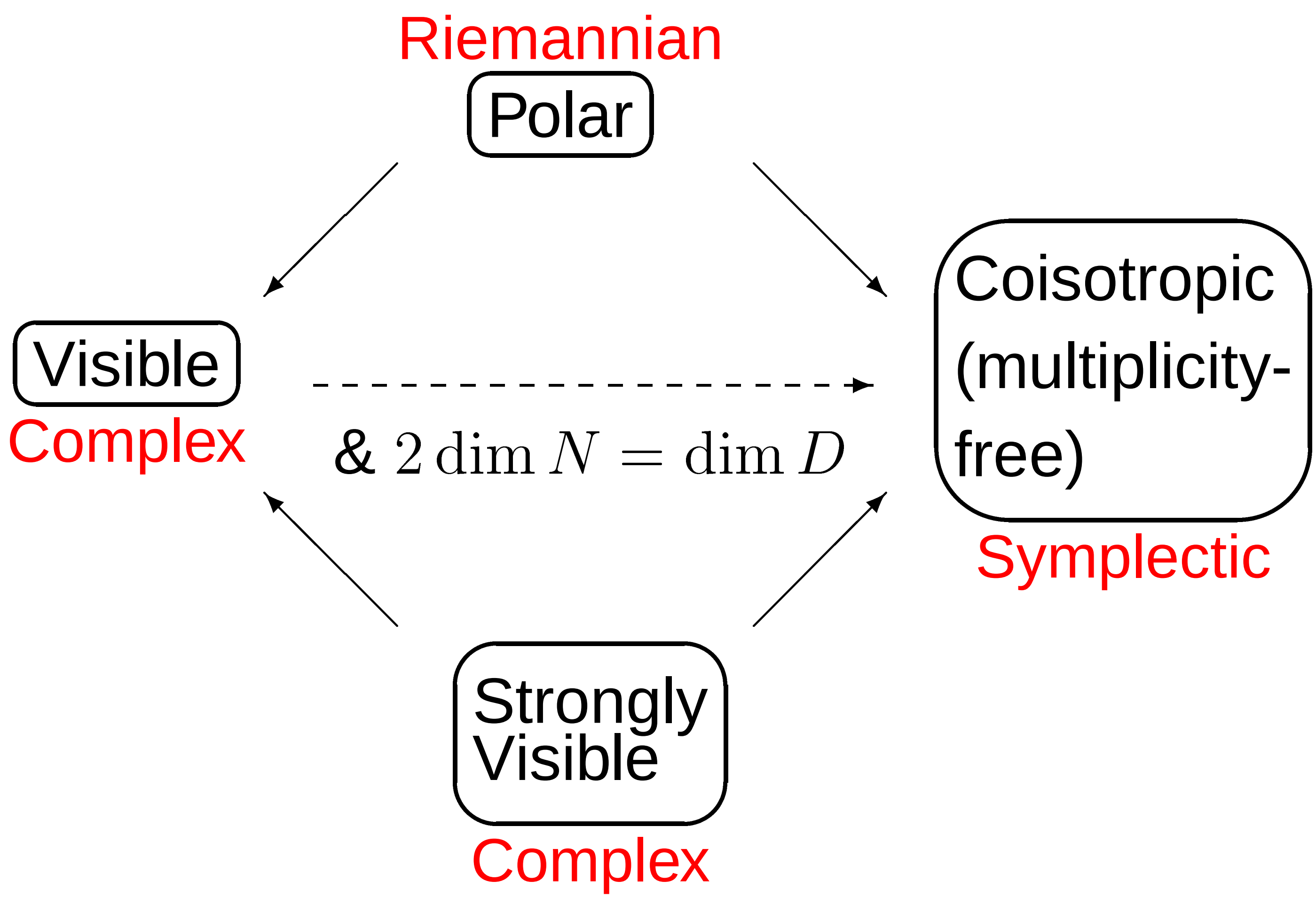

G/K Hermitian symmetric space

<u>Ex.</u>

$$G = SL(2,\mathbb{R})$$
$$K = SO(2)$$
$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}$$
$$N = \{ \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) : x \in \mathbb{R} \}$$
$$G/K \simeq \{ z \in \mathbb{C} : |z| < 1 \}$$

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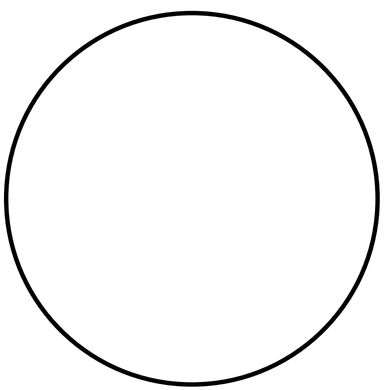

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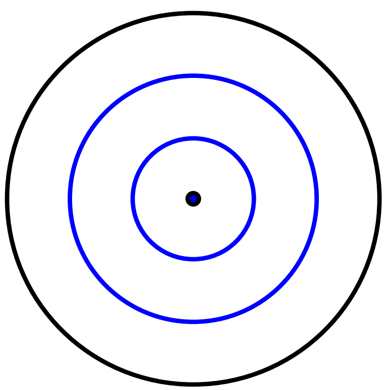

K-orbits

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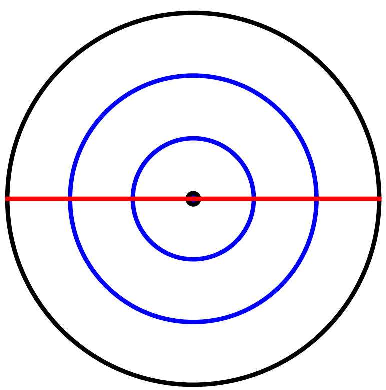

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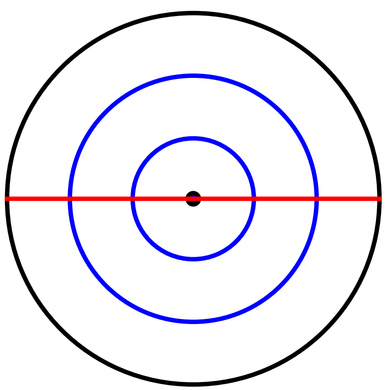

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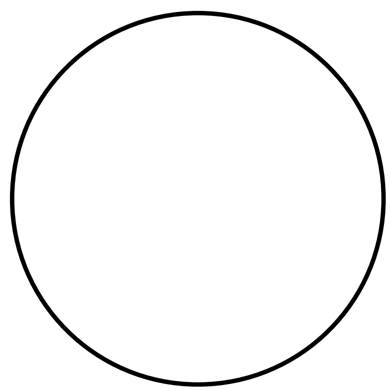

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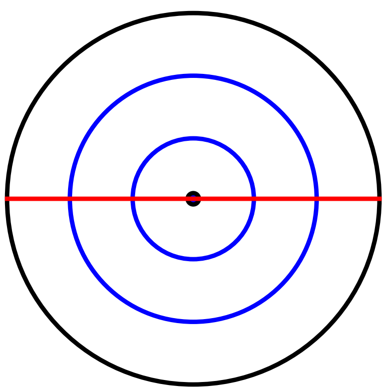

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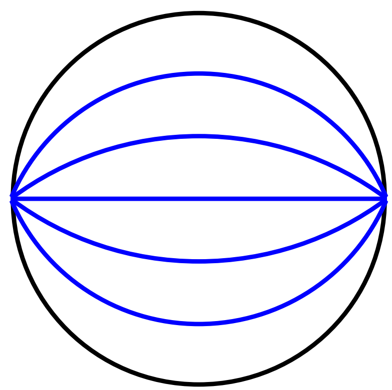

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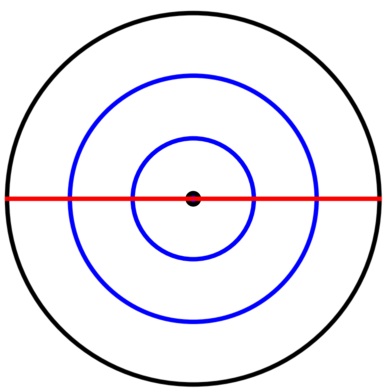

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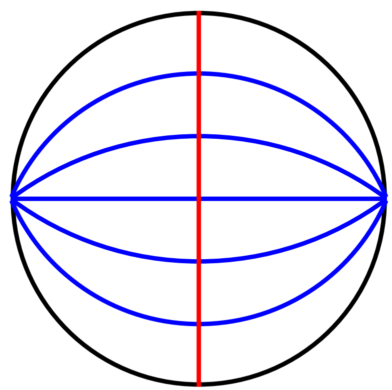

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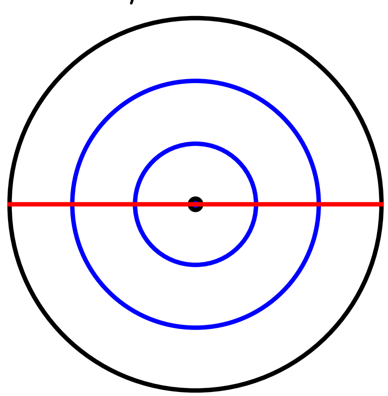

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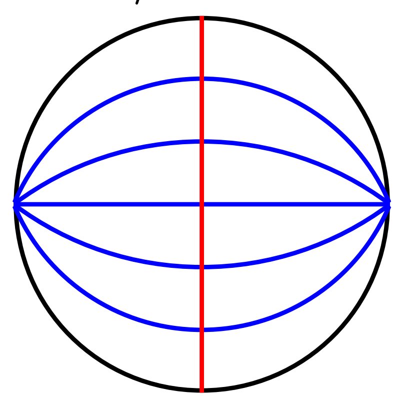

H-orbits

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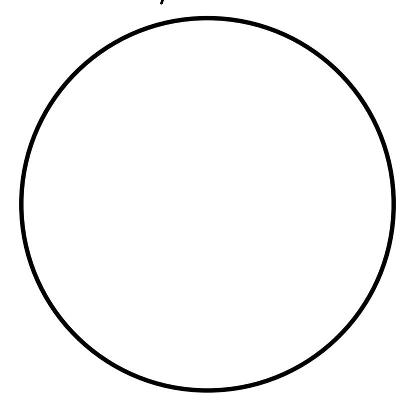

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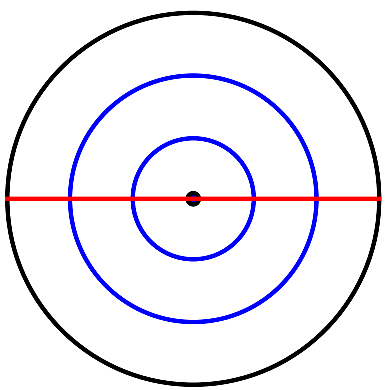

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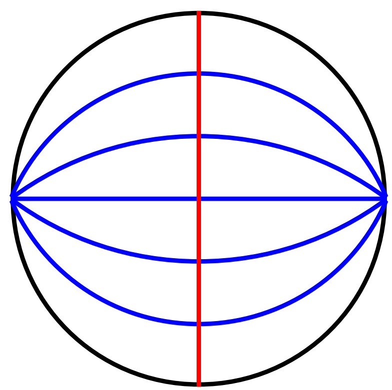

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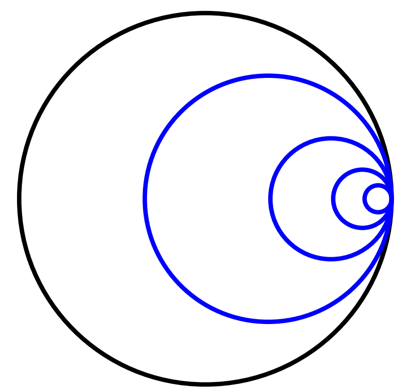

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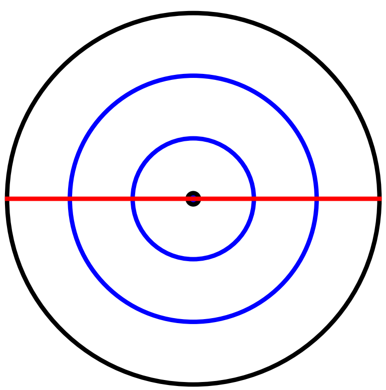

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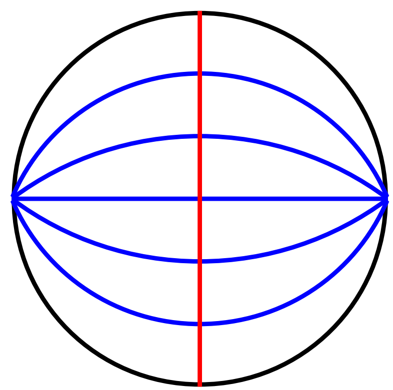

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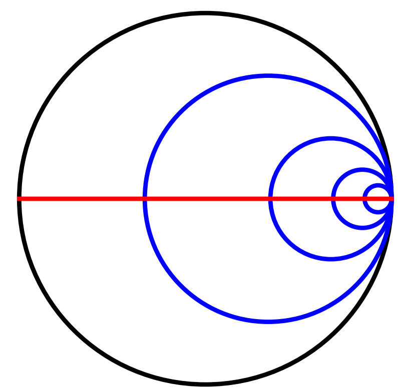

N-orbits

Proof (final step)

Theorem (math.DG/0607005, to appear in Transformation Group)

Assume $\begin{cases} G/K & \text{Hermitian symm. sp.} \\ (G,H) & \text{symmetric pair} \end{cases}$

$\Longrightarrow H \curvearrowright G/K$ is (strongly) visible

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$\Downarrow$ Propagation Theorem

Thm A $\pi_\lambda \otimes \pi_\mu, \pi_\lambda|_H$ are multiplicity-free.

π_λ: gen rectangular-shaped rep / highest wt rep of scalar type

Learning from G/K

$$\begin{array}{ccccc} G/K & & G \downarrow K & & \\ & \searrow & \vdots & & \pi \in \widehat{G} \\ G/H & \cdots\cdots & G \downarrow H & & \end{array}$$

<u>Problem</u> Find a theory for $G \downarrow H$
as a counterpart to the following G/K result.

- $L^2(G/K)$ is multiplicity-free $\Rightarrow$ Thm A
- G compact: Cartan–Helgason formula $\Rightarrow$?
- G non-compact: no discrete series for G/K $\Rightarrow$?

Visible actions on symmetric space

$$\boxed{\begin{array}{cc} G/K & G \downarrow K \\ G/H & G \downarrow H \end{array}} \qquad \pi_\lambda \in \widehat{G}$$

Visible actions on symmetric space

G/K	$G \downarrow K$
G/H	$G \downarrow H$

$\pi_\lambda \in \widehat{G}$

Fact (É. Cartan '29, Gelfand '50) $L^2(G/K)$ is multiplicity-free.

Thm A $\pi_\lambda|_H$ is multiplicity-free
if π_λ is gen. rectangular-shaped rep / scalar highest wt rep

Visible actions on symmetric space

G/K	$G \downarrow K$
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$\pi_\lambda \in \widehat{G}$

Fact (É. Cartan '29, Gelfand '50) $L^2(G/K)$ is multiplicity-free.

$\Uparrow$ propagation of MF from trivial rep

Thm A $\pi_\lambda|_H$ is multiplicity-free
if π_λ is gen. rectangular-shaped rep / scalar highest wt rep

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Visible actions on symmetric space

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$\Uparrow$ propagation of MF from trivial rep

$G \curvearrowright G_{\mathbb{C}}/K_{\mathbb{C}}$ strongly visible

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$\Uparrow$ propagation of MF from one dim'l rep

$H \curvearrowright G/K$ (Hermitian symm) strongly visible

Explicit decomposition formula

G/K	$G \downarrow K$	$\pi_\lambda \in \widehat{G}$
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Cartan–Helgason

(G: compact)

$$L^2(G/K) \simeq \sum_{\mu \in \Lambda}^{\oplus} \pi_\mu^G$$

Theorem B

Hua–Schmid–K–

$$\pi_\lambda^G|_H \simeq \sum_{\mu \in \Lambda'}^{\oplus} \pi_{\lambda-\mu}^H$$

$\operatorname{rank} \Lambda = \mathbb{R}\text{-rank}\, G$ $\qquad$ $\operatorname{rank} \Lambda' = \mathbb{R}\text{-rank}\, G/H$

free Abelian semigp

Holomorphic involution

Def. (G, H) symmetric pair, holomorphic type

$\Longleftrightarrow G/K$: Hermitian symm sp of non-compact type

$$\begin{cases} (G^\tau)_0 \subset H \subset G^\tau \\ \tau \curvearrowright G/K \quad \text{holomorphic} \end{cases}$$

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Ex. $G = Sp(n, \mathbb{R}) \quad (n = p + q)$

holomorphic $\quad H = \begin{cases} Sp(p, \mathbb{R}) \times Sp(q, \mathbb{R}) \\ U(p, q) \\ U(n) \ (= K) \end{cases}$

anti-holomorphic $\quad H = GL(n, \mathbb{R})$

(relative) strongly orth. roots

Def. (G, H) symmetric pair, holomorphic type

$\iff {}^{\exists}\tau \in \mathrm{Aut}(G),\ \tau^2 = \mathrm{id}$ s.t.

$$\begin{cases} (G^\tau)_0 \subset H \subset G^\tau \\ \tau \curvearrowright G/K \quad \text{holomorphic} \end{cases}$$

$$\begin{array}{ccccccccc} & & \mathfrak{g}_{\mathbb{C}} & = & \mathfrak{k}_{\mathbb{C}} & + & \mathfrak{p}_+ & + & \mathfrak{p}_- \\ & & \cup & & \cup & & \cup & & \cup \\ \mathfrak{h}_{\mathbb{C}} & = & \mathfrak{g}_{\mathbb{C}}^\tau & = & \mathfrak{k}_{\mathbb{C}}^\tau & + & \mathfrak{p}_+^\tau & + & \mathfrak{p}_-^\tau \end{array}$$

$$\begin{array}{ccccc} & \mathfrak{t}^\tau & \subset & \mathfrak{t} & \\ \text{Cartan} & \cap & & \cap & \text{Cartan} \\ & \mathfrak{k}^\tau & \subset & \mathfrak{k} & \end{array}$$

$\{\nu_1, \ldots, \nu_k\}$ maximal set of strongly orth. roots in $\Delta(\mathfrak{p}_+^{-\tau}, \mathfrak{t}^\tau)$

Note: $k = \mathbb{R}\text{-rank}\, G/H$ (cf. Koranyi–Wolf, $H = K$ case)

Discretely decomposable cases

Thm B (math.RT/0607002, to appear in Progr Math)

$$\begin{cases} \pi^G(\mu) \in \hat{G} & \text{holo. disc. series, scalar type} \\ (G, H) & \text{symmetric pair, holomorphic type} \end{cases}$$

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$H = K \cdots$ Hua–Kostant–Schmid

$G = SU(2,2) \cdots$ Jakobsen–Vergne

Explicit decomposition formula

G/K $\quad G \downarrow K$

G/H $\quad G \downarrow H$ $\qquad \pi_\lambda \in \widehat{G}$

$L^2(G/K)$ MF $\quad \cdots \quad$ $\pi_\lambda|_H$ MF

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$\Downarrow$ 'perturbation' $\Downarrow$

uniform estimate of support
and uniformly bounded multiplicities

(K– , Invent '94) (K– , to appear in Progr Math)

$\Downarrow$ wave front set

existence/non-existence
of continuous spectrum
for restriction $\pi|_H$

Learning from G/K

$$\begin{array}{ccccc} G/K & & G \downarrow K & & \\ & \searrow & \vdots & & \pi \in \widehat{G} \\ G/H & \cdots\cdots & G \downarrow H & & \end{array}$$

<u>Problem</u> Find a theory for $G \downarrow H$
as a counterpart to the following G/K result.

- $L^2(G/K)$ is multiplicity-free $\Rightarrow$ Thm A
- G compact: Cartan–Helgason formula $\Rightarrow$ Thm B
- G non-compact: no discrete series for G/K $\Rightarrow$?

Exclusive Law on discrete spectrum

G/K		$G \downarrow K$	$\pi \in \widehat{G}$
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Property of $\pi \in \widehat{G}$

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Theorem C (Exclusive Law)

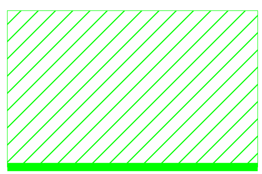
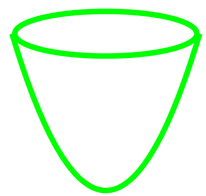
Let (G, H) be non-compact semisimple symmetric pair.

Then, ① $\Rightarrow$ not ②

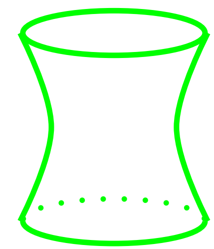
② $\Rightarrow$ not ①

$G = SL(2,\mathbb{R})$ $\quad K = SO(2)$ $\quad H = SO(1,1)$

$K \simeq S^1$ $\quad H \simeq \mathbb{R}$

 $\simeq$ 

G/K



G/H

$\mathrm{Disc}(G/K) = \emptyset$ $\quad \mathrm{Disc}(G/H) \neq \emptyset$

$\Uparrow$ $\quad \pi \in \widehat{SL(2,\mathbb{R})}$ $\quad \Downarrow$

$\pi|_K$ is always discretely decomposable

$\pi|_H$ is <u>not</u> discretely decomposable ($^\exists$continuous spectrum)

What does ‘Exclusive Law’ mean?

G/K	$G \downarrow K$	$\pi \in \widehat{G}$
G/H	$G \downarrow H$	

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Thm C (Exclusive Law) For $\pi \in \widehat{G}$, ① $\Rightarrow$ not ②, ② $\Rightarrow$ not ①.

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Ex.1 $(G/K \longleftrightarrow G \downarrow K)$

$G \downarrow K$ is obviously discretely decomposable

$\Longrightarrow G/K$ has no discrete series

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Thm C (Exclusive Law) For $\pi \in \widehat{G}$, ① $\Rightarrow$ not ②, ② $\Rightarrow$ not ①.

Ex.2 G/H has holomorphic discrete series rep π

(i.e. $\pi =$ highest wt rep occuring in $L^2(G/H)$)

$\iff H/H \cap K \subset G/K$ totally real.

Proof $\Longleftarrow$ Construction (Ólafsson–Ørsted ’88, K–)

$\Longrightarrow$ Use Thm C.

$\pi|_H$ cannot be discretely decomposable.

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Ex.3 vanishing/non-vanishing type thm of modular varieties

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$\Gamma \subset G$ torsion-free, cocompact discrete subgp

s.t. $\Gamma \cap H \subset H$ is also cocompact

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modular variety

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modular variety

Pairing of $\iota(Y)$ against π-component in $H^{\dim Y}_{\text{de Rham}}(X, \mathbb{C})$

(Matsushima–Murakami, Borel–Wallach, Parthasarathy, Vogan–Zuckerman)

- ① $\Rightarrow$ non-vanishing type thm (Tong–Wang '89, Speh '07)

$\Updownarrow$ Exclusive Law (Thm C)

- ② $\Rightarrow$ vanishing type thm (K–Oda)

Proof outline of Thm C

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- discrete series for G/H
  (Flensted-Jensen, Matsuki–Oshima, late 70s to 80s)
- derived functor $(\mathfrak{g}, K)$-module
  (Zuckerman, Vogan, late 70s to 80s)
- Discrete decomposability of restriction of unitary reps
  (K– , 90s) (Invent '94, Ann Math '98, Invent '98)
  miclolocal analysis, associated varieties,
  and "perturbation" of Cartan–Helgason thm

Learning from G/K

G/K		$G \downarrow K$	
	$\searrow$	$\vdots$	$\pi \in \widehat{G}$
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as a counterpart to the following G/K result.

- $L^2(G/K)$ is multiplicity-free $\underset{\text{visible action}}{\Longrightarrow}$ Thm A
- G compact: Cartan–Helgason formula $\Longrightarrow$ Thm B perturbation
- G non-compact: no discrete series for G/K $\Longrightarrow$ Thm C