An Index Theorem for Wiener–Hopf Operators

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Initial Problem

G/H : ordered symmetric space (noncompactly causal), P : causal semigroup, $W_f : L^2(P) \to L^2(P)$.

Problem

Solve $(1 + W_f)u = v$ *(i.e., invent Fredholmness)*

Framework for today's talk

General setup

Given (closed pointed solid) convex cone $\Omega \subset \mathbb{R}^n$: i) investigate the integral equation $(1 + W_f)u = v$, where

$$W_f u(x) = \int_\Omega f(x-y)u(y)\,dy, \quad f \in L^1(\mathbb{R}^n), u \in L^2(\Omega), x \in \Omega$$

ii) the C*-algebra $\mathrm{C}^*(WH) = \langle W_f \rangle$.

Some historical results

Gohberg/Krein ('58):	$\Omega = (0, \infty)$
Berger/Coburn/Douglas,... ('69–∼'80):	Quarter half-plane
Muhly/Renault ('82):	Polyhedral or symm. self-dual
Upmeier ('84–'88):	Symmetric self-dual cones.
Hilgert/Neeb ('95):	Ordered symmetric spaces.

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Order compactification of $\Omega \subset \mathbb{R}^n$

- $\mathbb{F}(\mathbb{R}^n)$: closed subsets of $\mathbb{R}^n$, topology: conv. seqs $(A_k) \subset \mathbb{F}(\mathbb{R}^n)$ with $\limsup A_k = \liminf A_k$
- $\eta : \mathbb{R}^n \hookrightarrow \mathbb{F}(\mathbb{R}^n)$, $x \mapsto x - \Omega$ – homeo. onto image
- $\overline{\mathbb{R}^n} := \overline{\eta(\mathbb{R}^n)}$, same for Ω (order compactification)

$\mathbb{R}^n$ acts from the right on $\overline{\mathbb{R}^n}$ by $A.x = A + x$ and fixes $\overline{\mathbb{R}^n}$ $\Longrightarrow$ so we form

$$\mathcal{W}_\Omega = (\overline{\mathbb{R}^n} \rtimes \mathbb{R}^n)|_{\overline{\Omega}}.$$

$$= \{(x - F^*, y_1 + y_2 - x) \mid x, y_1 \in \langle F \rangle \cap F^*, y_2 \in F^\perp, F \in P\}$$

Composition: $((x, g), (y, h)) \mapsto (x, g)(y, h) = (x, gh)$ if $x.g = y$

Range map $r : \cdots \mapsto x - F^*$ Source map $s : \cdots \mapsto y_1 - F^*$

Units: $\mathcal{W}_\Omega^{(0)} = \overline{\Omega}$.

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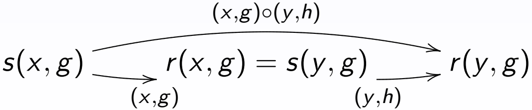

Local bundle structure of $\overline{\Omega}$

Increasing ordering of face dimensions

$$\{0 = n_0 < n_1 < \ldots < n_d + n\} = \{\dim F \,|\, F \subset \Omega^* \text{ convex face}\},$$

- P_j: set of n_{d-j}-dim. faces of Ω^* – ASSUME compact for all j (polyhedral: all P_j finite; symm. self-dual: P_j compact homo. space)
- $\exists$surjection from $\overline{\Omega} = \mathcal{W}^0_\Omega$ *onto* set of all faces of Ω^* whose restriction to the inverse – denoted Y_j – is continuous. Then Y_j is closed invariant.
- $U_j := \bigcup_{i=0}^{j-1} Y_i$ open invariant.
- $\Sigma_j = \mathcal{W}_\Omega|_{P_j} \to P_j$ is an oriented real vector bundle of rank $n - n_{d-j}$ (complicated; uses a lot of geometric measure theory).

The Wiener–Hopf Groupoid C*-algebra $C^*(\mathcal{W}_\Omega)$

Same as for locally compact groups G, only more complicated... integrate irreducible representations of $\mathcal{W}_\Omega$ first to the Banach $*$-algebraic bundle $L^1(\mathcal{W}_\Omega)$ and then C*-complete to get $C^*(\mathcal{W}_\Omega)$.

Concretely: For $(F, y) \in \Sigma_j$, $\varphi \in C_c(\mathcal{W}_\Omega)$, $h \in L^2(\langle F \rangle \cap F^*)$, and $v \in \langle F \rangle \cap F^*$, one defines

$$L^{F,y}(\varphi)h(v) = \int_{F^\perp} \int_{\langle F \rangle \cap F^*} \varphi(F, v, w_1 + w_2 - v) e^{-2\pi i (w_2, y)} h(w_1)\, dw_1\, dw_2.$$

These yield all irr. reps. of $C^*(\mathcal{W}_\Omega)$.

Theorem (Muhly–Renault, '82)

$C^*(WH) \simeq C^*(\mathcal{W}_\Omega)$.

Problem: Ordered symmetric spaces

G/H ordered symmetric space, P causal semigroup, $W_f : L^2(P) \to L^2(P)$. G **not** amenable $\Longrightarrow$ $C^*(\mathcal{W}_\Omega) \to C^*(WH)$ not injective.

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Spectrum of $C^*(\mathcal{W}_\Omega)$

Theorem (Muhly–Renault ('82), Alldridge–J ('06))

The sets $U_j \subset \overline{\Omega}$, $j = 0, \dots, d+1$, form an ascending chain of open invariant subsets. The ideals $I_j = C^*(\mathcal{W}_\Omega | U_j)$ form a composition series with liminary quotients $I_{j+1}/I_j = C^*(\mathcal{W}_\Omega | Y_j)$. Hence, the C^*-algebra $C^*(\mathcal{W}_\Omega)$ is of type I. Its spectrum is the set $\Sigma = \bigcup_{j=0}^{d} \Sigma_j$, the topology of which is given by the sets $U \cup \bigcup_{i=0}^{j-1} \Sigma_i$ for all $0 \le j \le d$ and all open $U \subset \Sigma_j$.

Definition

An element $a \in C^*(\mathcal{W}_\Omega)$ is *j-Fredholm* if it is invertible in the unitization of $C^*(\mathcal{W}_\Omega)/I_j$.

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Analytical Indices

Warning: abstract nonsense

The C*-algebra extension

$$0 \longrightarrow C^*(\mathcal{W}_\Omega|_{Y_{j-1}}) \longrightarrow C^*(\mathcal{W}_\Omega|_{U_{j+1}\setminus U_j}) \xrightarrow{\sigma} C^*(\mathcal{W}_\Omega|_{Y_j}) \longrightarrow 0$$

gives exact hexagon

$$\begin{array}{ccccc} K_c^1(\Sigma_{j-1}) & \longrightarrow & K_c^1(\cdots) & \longrightarrow & K_c^1(\Sigma_j) \\ \uparrow \partial_j & & & & \downarrow \partial_j \\ K_c^0(\Sigma_j) & \longleftarrow & K_c^0(\cdots) & \longleftarrow & K_c^0(\Sigma_{j-1}) \end{array}$$

We handle such extensions abstractly with Kasparov's KK-theory, which expresses ∂_j (very abstractly) in terms of the Kasparov product. The abstract symbol maps σ_j have a comparably shady origin. . .

Analytical indices

Example (Gohberg–Krein – $\Omega = [0, \infty)$)

$0 \longrightarrow \mathcal{K} \longrightarrow C^*(WH) \xrightarrow{\sigma} C_0(\mathbb{R}) \longrightarrow 0$ where $\sigma(W_f) = \widehat{f}$, and $1 + W_f$ is Fredholm $\iff 1 + \widehat{f} \neq 0$ on $\mathbb{R}$ ($\partial([\sigma(T)]) = \operatorname{Index} T$).

Example (Classical example – Atiayh–Singer 101)

M compact C^∞-manifold, P ΨDO (order 0, say), σ its principal symbol map: $0 \longrightarrow \mathcal{K}(L^2(M)) \longrightarrow \Psi^0(M) \xrightarrow{\sigma} C(\mathbb{S}^*M) \longrightarrow 0$ gives $\partial : K^1(\mathbb{S}^*M) \to K_0(\mathcal{K}) \simeq \mathbb{Z}$ with $\operatorname{index} P = \partial([\sigma(P)])$.
Even ok for families of elliptic ΨDO's acting on vector bundles.

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Analytical Index Formula

Theorem (Alldridge-J, '06)

The quotient I_{j+1}/I_j is a field $\mathcal{K}_j$ of elementary C^*-algebras over Σ_j. If a class $f \in K^1_c(\Sigma_j)$ is represented by an element invertible modulo matrices over I_j, then its image σ_j in the matrices over $M(\mathcal{K}_j)$ is a Fredholm family, and

$$\partial_j(f) = Index_{\Sigma_{j-1}} \sigma_j(f)$$

is the analytical family index.

Essential ingredients in proof:

1. If a is j-Fredholm, then $\sigma_{j-1}(a) = (L^{F,y}(a))_{(F,y)\in\Sigma_{j-1}}$ is a continuous family of Fredholm operators on $L^2(\langle F\rangle \cap F^*)$
2. The family index makes sense (Σ_{j-1} not compact!)

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Topological underpinnings

$$\mathcal{P}_j = \{(E,F) \in P_{j-1} \times P_j \,|\, F \subset E\}$$

$$P_{j-1} \xleftarrow{\ \xi\ } \mathcal{P}_j \xrightarrow{\ \eta\ } P_j$$

Then:

- $\mathcal{P}_j \to \xi(\mathcal{P}_j) \Subset P_{j-1}$ is a fibrewise C^1-manifold
- $\eta^*\Sigma_j$ is the trivial bundle over $\xi^*\Sigma_{j-1} \oplus T\mathcal{P}_j$, where $T\mathcal{P}_j$ is the fibrewise tangent bundle.

Index Theorem

Topological Index Theorem (Alldridge-J, '06)

The KK-theory element ∂_j representing the jth Wiener–Hopf extension is given by $\zeta_* \partial_j = \boldsymbol{\eta}^*[y \otimes \tau_j]$ where $y \in KK^1(\mathbb{C}, S)$ represents the classical Wiener–Hopf extension, $\boldsymbol{\eta}$ is associated to the projection $\eta^*\Sigma_j \to \Sigma_j$, and $\boldsymbol{\zeta}$ is associated to the inclusion $\Sigma_{j-1}|\xi(\mathcal{P}_j) \subset \Sigma_{j-1}$. Here,

$$\tau_j \in KK(C_r^*(T\mathcal{P}_j \oplus \xi^*\Sigma_{j-1}), C_r^*(\Sigma_{j-1}|\xi(\mathcal{P}_j)))$$

(the generalized Connes–Skandalis map associated to the tangent groupoid of $\Sigma_{j-1}|\xi(\mathcal{P}_j)$) represents the Atiyah–Singer family index for $T\mathcal{P}_j \oplus \xi^*\Sigma_{j-1}$, considered as a vector bundle over $\Sigma_{j-1}|\xi(\mathcal{P}_j)$.

Ingredients in proof

- Non-commutative geometry (extends Connes, Hilsum–Skandalis, ...)
- Algebraic topology, convex geometry and geometric measure theory
- Lipschitz embedding of fibres in $\mathcal{C}^{1,0}$-continuous groupoid bundles.

References

- Alldridge/Johansen: *Spectrum and analytical indices of the C^*-algebra of Wiener–Hopf operators*, J. Funct. Anal. **249** (2007), 425–452.
- — : *An index formula for Wiener–Hopf operators*, arXiv: `math.OA/0611198` (submitted)