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Infinite dimensional spherical analysis

The spherical analysis of Gelfand pairs plays an important role in the analysis on Riemannian symmetric spaces. It involves geometry, Fourier analysis, and special functions. The Gelfand pair theory occurs also for further homogeneous spaces as, for instance, homogeneous graphs.

Consider an increasing sequence of Gelfand pairs $\big(G(n), K(n)\big)$. Analysis on the inductive limit (G, K):

$$G = \bigcup_{n=1}^{\infty} G(n), \quad K = \bigcup_{n=1}^{\infty} K(n),$$

has been developped by Olshanski. This analysis belongs to asymptotic harmonic analysis. In fact, in this analysis, one studies asymptotics of functions or measures on $G(n)$ as $n \to \infty$. For instance the spherical functions for the pair (G, K) are limits of spherical functions for $\big(G(n), K(n)\big)$. This a frame where various results in infinite dimensional harmonic analysis take place:

- on the infinite dimensional unitary group by Voiculescu, Kerov and Vershik,
- on the space of infinite dimensional Hermitian matrices by Pickrell, Olshanski and Vershik,
- on the infinite symmetric group by Thoma, Kerov and Vershik.

Spherical analysis on Gelfand pairs

We recall first the definition of a Gelfand pair and some classical results. Let G be a locally compact group and K a compact subgroup. Assume that (G, K) is a Gelfand pair. It means that the convolution algebra $L^1(K\backslash G/K)$ of K-biinvariant integrable functions is commutative. A spherical function is a continuous function φ on G which is K-biinvariant and satisfies

$$\int_K \varphi(xky)\alpha(dk) = \varphi(x)\varphi(y) \quad (x, y \in G),$$

where α denotes the normalized Haar measure on K. The characters of the commutative Banach algebra $L^1(K\backslash G/K)$ are given by bounded spherical functions: such a character has the form

$$\chi(f) = \int_G f(x)\varphi(x)dx,$$

where φ is a bounded spherical function, and dx is a Haar measure on G (which is unimodular).

The property for (G, K) to be a Gelfand pair is reflected in representation theory. Let π be a unitary representation of G on a Hilbert space $\mathcal{H}$. Assume that there is in $\mathcal{H}$ a K-invariant cyclic vector. Then the representation π is multiplicity free, that is: the commutant $\pi(G)'$ is a commutative algebra,

Let $\mathfrak{P}(K\backslash G/K)$ be the cone of K-biinvariant continuous functions φ on G which are of positive type, and

- $\mathfrak{P}_1(K\backslash G/K)$ with $\varphi(e) = 1$ (e is the identity element of G),
- $\mathfrak{P}_{\leq 1}(K\backslash G/K)$ with $\varphi(e) \leq 1$.

By the Gelfand-Naimark-Segal construction, for $\varphi \in \mathfrak{P}_1(K\backslash G/K)$, one obtains a unitary representation $(\pi^\varphi, \mathcal{H}^\varphi)$ with a K-invariant cyclic vector $u \in \mathcal{H}^\varphi$ such that

$$\varphi(x) = \big(\pi^\varphi(x)u|u\big).$$

Then, for $\varphi \in \mathfrak{P}_1(K\backslash G/K)$, the following properties are equivalent.

- The function φ is spherical.

- The function φ is an extremal point in the convex set $\mathfrak{P}_1(K\backslash G/K)$.

- The representation π^φ is irreducible.

Let Ω denote the set of spherical functions of positive type. With the topology of uniform convergence on compact sets in G, the set Ω is locally compact. There is in this setting an analogue of Bochner's Theorem.

THEOREM (BOCHNER-GODEMENT). — *For every function $\varphi \in \mathfrak{P}(K\backslash G/K)$, there is a unique positive bounded measure μ on Ω such that*

$$\varphi(x) = \int_\Omega \omega(x)\mu(d\omega).$$

This integral representation can be proven by using Krein-Milman Theorem. It is related to the decomposition of the representation π as direct integral of irreducible ones.

Spherical pairs of Olshanski

We consider now an increasing sequence $\big(G(n), K(n)\big)$ of Gelfand pairs.

- $G(n)$ is a closed subgroup of $G(n+1)$,
- $K(n)$ is a closed subgroup of $K(n+1)$,
- $K(n) = G(n) \cap K(n+1)$,

Define

$$G = \bigcup_{n=1}^{\infty} G(n), \quad K = \bigcup_{n=1}^{\infty} K(n).$$

A spherical function is a continuous function φ on G which is K-biinvariant and satisfies

$$\lim_{n \to \infty} \int_{K(n)} \varphi(xky)\alpha_n(dk) = \varphi(x)\varphi(y),$$

where α_n is the normalized Haar measure of $K(n)$.

In general the group G is not locally compact, hence there is no Haar measure on G. Therefore there is no obvious analogue of the convolution algebra of K-biinvariant integrable functions on G. However the basic facts in representation theory for Gelfand pairs extend to Olshanski spherical pairs.

For $\varphi \in \mathfrak{P}_1(K \backslash G/K)$, let π^{φ} be the associated unitary representation by the Gelfand-Naimark-Segal construction. The following equivalence still holds:

- The function φ is spherical.

- The function φ is an extremal point in the convex set $\mathfrak{P}_1(K \backslash G/K)$.

- The representation π^{φ} is irreducible.

For a given Olshanski spherical pair, a basic problem is to determine the set Ω of spherical functions of positive type. This has been solved for a number of special cases. A new phenomenon shows up: in number of cases the spherical functions have a multiplicativity property.

Space of infinite dimensional Hermitian matrices

The unitary group $U(n)$ acts on the space $H(n) = Herm(n, \mathbb{C})$ as follows:

$$x \mapsto uxu^* \quad \bigl(x \in H(n), u \in U(n)\bigr).$$

Define

$$G(n) = U(n) \ltimes H(n),\ K(n) = U(n).$$

A K-biinvariant function on G can be seen as a function on the space $H(\infty)$ consisting in infinite dimensional Hermitian matrices $x = (x_{ij})$ such that $x_{ij} = 0$ for i, j large enough. A spherical function φ has the following form

$$\varphi(x) = \det \Phi(x),$$

where Φ is a continuous function on $\mathbb{R}$ with $\Phi(0) = 1$. It means that

$$\varphi\bigl(\mathrm{diag}(z_1, \ldots, z_n, 0, \ldots)\bigr) = \Phi(z_1) \ldots \Phi(z_n).$$

We make a definition: We say that a continuous function Φ on $\mathbb{R}$ with $\Phi(0) = 1$ is a *Pólya function* if, for every n, the function

$$\varphi(x) = \det \Phi(x)$$

is of positive type on $H(n)$. We can state: The spherical function φ is of positive type if and only if Φ is a Pólya function.

THEOREM (PICKRELL, 1991, OLSHANSKI-VERSHIK, 1996). *The Pólya functions are the following ones:*

$$\Phi(z) = e^{-i\beta z} e^{-\frac{1}{2}\gamma z^2} \prod_{k=1}^{\infty} \frac{e^{i\alpha_k z}}{1 + i\alpha_k z},$$

with

$$\beta \in \mathbb{R},\ \gamma \geq 0,\ \alpha_k \in \mathbb{R},\ \sum_{k=1}^{\infty} \alpha_k^2 < \infty.$$

First method of proof (Pickrell)

Assume that Φ is the Fourier transform of an integrable function f on $\mathbb{R}$. Then Φ is a Pólya function if and ond only if the function f is totally positive: for all numbers $s_1 < \cdots < s_n$, $t_1 < \cdots < t_n$,

$$\det\Big(\big(f(s_i - t_j)\big)_{1 \leq i,j \leq n}\Big) \geq 0.$$

By a Theorem of Schoenberg (1951), the Fourier transform of an integrable totally positive function has the above form.

Second method (Olshanski-Vershik)

By a result fo Vershik, every spherical function φ for the Olshanski spherical pair (G, K) is the limit of a sequence $\varphi^{(n)}$, where $\varphi^{(n)}$ is a spherical function for the Gelfand pair $\big(G(n), K(n)\big)$. A spherical function for the pair $\big(G(n), K(n)\big)$ is the Fourier transform of an orbital measure for the action of the unitary group $U(n)$ on the space $H(n)$:

$$\varphi_\lambda^{(n)}(x) = \int_{U(n)} e^{i\,\mathrm{tr}(xu\lambda u^*)} \alpha(du),$$

where $\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, and α is the normalized Haar measure of $U(n)$. Consider the following power expansion:

$$\varphi_\lambda^{(n)}\big(\mathrm{diag}(z, 0, \ldots, 0)\big) = \sum_{m=0}^{\infty} a_m^{(n)}(\lambda) z^m.$$

For suitable sequences $\lambda(n)$, the coefficients have limits:

$$\lim_{n\to\infty} a_m^{(n)}\big(\lambda(n)\big) = a_m,$$

and

$$\sum_{m=0}^{\infty} a_m z^m = e^{-i\beta z} e^{-\frac{1}{2}\gamma z^2} \prod_{k=1}^{\infty} \frac{e^{i\alpha_k z}}{1 + i\alpha_k z}.$$

The set Ω of spherical functions of positive type can be identified with the set of the parameters (α, β, γ) submitted to the above conditions. One considers on Ω the topology of uniform convergence on compact sets. This topology can be described in terms of the parameters.

$$\Omega \equiv \mathbb{R} \times \Omega_0,$$

$$\Omega_0 \equiv \{(\alpha, \gamma) \mid \gamma \geq 0,\ \alpha_k \in \mathbb{R},\ \sum_{k=1}^{\infty} \alpha_k^2 < \infty\}.$$

To an element $\omega = (\alpha, \gamma) \in \Omega_0$ one associates the measure σ on $\mathbb{R}$ such that

$$\int f(t)\sigma(dt) = \gamma f(0) + \sum_{k=1}^{\infty} \alpha_k^2 f(\alpha_k).$$

We equip Ω_0 with the topology of weak convergence for the measures σ. Then Ω is homeomorphic to $\mathbb{R} \times \Omega_0$.

It has been proven by Bouali (2006) that previous Theorem extends to the case of $H(n) = Herm(n, \mathbb{F})$, with $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Then Pólya functions are the following ones

$$\Phi(z) = e^{-i\beta z} e^{-\frac{1}{2}\gamma z^2} \prod_{k=1}^{\infty} \frac{e^{i\alpha_k z}}{\left(1 + i\frac{2}{d}\alpha_k z\right)^{\frac{d}{2}}},$$

where $d = \dim_{\mathbb{R}} \mathbb{F}$.

Infinite dimensional unitary group

Consider now

$$G(n) = U(n) \times U(n),\ K(n) = \mathrm{diag}\big(G(n)\big).$$

A K-invariant function on G can be seen as a central function on the infinite dimensional unitary group $U(\infty)$ consisting in infinite dimensional unitary matrices $u = (u_{ij})$ such that $u_{ij} = \delta_{ij}$ for i, j large enough. A spherical function φ has the following from

$$\varphi(u) = \det \Phi(u),$$

where Φ is a continuous function on the circle $\mathbb{U} = U(1)$ with $\Phi(1) = 1$. We say that a continuous function Φ on $\mathbb{U}$ is a *Voiculescu function* if, for every n, the function

$$\varphi(u) = \det \Phi(u)$$

is of positive type on $U(n)$. (Voiculescu says *une bonne fonction.*)

THEOREM (VOICULESCU, 1976; KEROV-VERSHIK, 1981; BOYER, 1983). — *The Voiculescu functions are the following ones:*

$$\Phi(z) = e^{\lambda(z-1)} e^{\mu(z^{-1}-1)} \prod_{k=1}^{\infty} \frac{1+\beta_k^+(z-1)}{1-\alpha_k^+(z-1)}\ \frac{1+\beta_k^-(z^{-1}-1)}{1-\alpha_k^-(z^{-1}-1)},$$

with

$$\lambda \geq 0,\ \mu \geq 0,\ \alpha_k^{\pm} \geq 0,\ \beta_k^{\pm} \geq 0,$$

and

$$\sum_{k=1}^{\infty} (\alpha_k^+ + \alpha_k^- + \beta_k^+ + \beta_k^-) < \infty.$$

First method of proof (Voiculescu, Boyer)

Consider the Fourier series expansion of Φ:

$$\Phi(z) = \sum_{m=-\infty}^{\infty} c_k z^k \quad (t \in \mathbb{U}).$$

Then Φ is a Voiculescu function if and only if the squence c_m is totally positive: for all $k_1 < \cdots < k_n$, $\ell_1 < \cdots < \ell_n$,

$$\det\Big(\big(c_{k_i - \ell_j}\big)_{1 \le i,j \le n}\Big) \ge 0.$$

By a theorem of Edrei (1953), extending a theorem of Schoenberg (1951), the generating function of a totally positive sequence has the above form.

Second method (Kerov-Vershik)

A spherical function for the pair $\big(G(n), K(n)\big)$ is a normalized character of the unitary group $U(n)$:

$$\varphi_\lambda^{(n)}(u) = \frac{\chi_\lambda^{(n)}(u)}{\chi_\lambda^{(n)}(\mathbf{1})}.$$

Consider the expansion

$$\varphi_\lambda^{(n)}\big(\mathrm{diag}(z, 1, \ldots, 1)\big) = \sum_{m=0}^{\infty} a_m^{(n)}(\lambda)(z-1)^m,$$

and look at sequences $\lambda(n)$ such that the coefficients have limits:

$$\lim_{n\to\infty} a_m^{(n)}\big(\lambda(n)\big) = a_m.$$

The second method has been extended to inductive limits of symmetric spaces of compact type.
(Okunkov-Olshanski, 1998, 1997).

In particular for
- $G(n) = U(n)$, $K(n) = O(n) \quad (d = 1)$,
- $G(n) = U(n) \times U(n)$, $K(n) = U(n) \quad (d = 2)$,
- $G(n) = U(2n)$, $K(n) = Sp(n) \quad (d = 4)$,

$G(n)/K(n)$ is the Shilov boundary of a Hermitian symmetric space of tube type of rank n. The spherical functions are Jack polynomials. Then the Voiculescu functions are the following ones:

$$\begin{aligned}\Phi(z) = &e^{\lambda(z-1)} e^{\mu(z^{-1}-1)} \\ &\prod_{k=1}^{\infty} \frac{1+\beta_k^+(z-1)}{\left(1-\frac{2}{d}\alpha_k^+(z-1)\right)^{\frac{d}{2}}} \ \frac{1+\beta_k^-(z^{-1}-1)}{\left(1-\frac{2}{d}\alpha_k^-(z^{-1}-1)\right)^{\frac{d}{2}}}.\end{aligned}$$

Bochner-Godement Theorem

An analogue of Bochner-Godement Theorem has been recently established by Rabaoui, by using the integral representation theory in convex cones due to Choquet.

Theorem (Rabaoui, 2007). — *For every $\varphi \in \mathfrak{P}(K\backslash G/K)$ there is a unique bounded positive measure μ on Ω such that*

$$\varphi(x) = \int_\Omega \omega(x)\mu(d\omega).$$

Unicity

One shows that the cone $\mathfrak{P}(K\backslash G/K)$ is a lattice.

Recall that a convex cone Γ is said to be a *lattice* if, for $\gamma_1, \gamma_2 \in \Gamma$, there exists a smallest common majorant in Γ (for the ordering defined by the cone Γ).

This is related to the commutativity property. One shows that, for every φ, the representation π^φ obtained from φ by the Gelfand-Naimark-Segal construction is multiplicity free: the commutant $\pi^\varphi(G)'$ is a commutative algebra. And this implies that the cone $\mathfrak{P}(K\backslash G/K)$ is a lattice.

Existence

In the case of a Gelfand pair one embeds $\mathfrak{P}(G)$ into $L^\infty(G)$ and considers on $L^\infty(G)$ the $*$-weak topology $\sigma\big(L^\infty(G), L^1(G)\big)$. The closed unit ball in $L^\infty(G)$ is compact for this topology.
A remarkable non trivial fact is that $\mathfrak{P}_{\leq 1}(G)$ is closed in the closed unit ball of $L^\infty(G)$ for that topology. It follows that the cap

$$\mathfrak{P}_{\leq 1}(G) = \{\varphi \in \mathfrak{P}(G) \mid \varphi(e) \leq 1\}$$

is compact.

But this does not hold anymore in general for an Olshanski spherical pair. The reason is as follows:

Let G be a locally compact group, H a closed subgroup. We consider on the cones $\mathfrak{P}(G)$ and $\mathfrak{P}(H)$ the $*$-weak topologies $\sigma\big(L^\infty(G)), L^1(G)\big)$, $\sigma\big(L^\infty(H)), L^1(H)\big)$. Then in general the restriction map

$$\mathrm{Res} \ : \ \mathfrak{P}(G) \to \mathfrak{P}(H),$$

is not continuous.

For instance take $G = \mathbb{R}$, $H = \{0\}$,

$$\varphi_n(x) = e^{inx}.$$

Then, for every n,

$$\varphi_n(0) = 1,$$

but, by Riemann-Lebesgue Lemma,

$$\lim_{n\to\infty} \varphi_n = 0,$$

for the topology $\sigma\big(L^\infty(\mathbb{R}), L^1(\mathbb{R})\big)$.

However the restriction map is lower semi-continuous.

Following an idea of Olshanski, one introduces the cone

$$\begin{aligned}\mathfrak{Q} =&\{\varphi = (\varphi^{(1)}, \varphi^{(2)}, \ldots) \mid \\ &\varphi^{(n)} \in \mathfrak{P}\big(K(n)\backslash G(n)/K(n)\big), \ \mathrm{Res}\,\varphi^{(n+1)} \ll \varphi^{(n)}\}.\end{aligned}$$

The cone $\mathfrak{Q}$ is closed in

$$\prod_{n=1}^{\infty} \mathfrak{P}\big(K(n)\backslash G(n)/K(n)\big),$$

and the cap $\mathfrak{Q}_{\leq 1}$ is compact. One can apply Choquet's Theorem to the cone $\mathfrak{Q}$.

Space of infinite dimensinal Hermitian matrices

D. Pickrel (1991). Mackey analysis of infinite classical motion groups, *Pacific J. Math.*, **150**, 139–166.

G. Olshanski, A. Vershik (1996). Ergodic unitarily invariant measures on the space of infinite Hermitian matrices, in *Comtemporary Mathematical Physics* (eds. R.L. Dobroshin, R.A. Minlos, M.A. Shubin, A.M. Vershik), *Amer. Math. Soc. Translations*, **175**, 137–175.

Infinite dimensional unitary group

D. Voiculescu (1976). Représentations factorielles de type II_1 de $U(\infty)$, *J. Math. Pures Appl.*, **55**, 1–20.

A. Vershik, S. Kerov (1982). Characters and factor representations of the infinite unitary group, *Soviet Math. Dokl.*, **26**, 570–574.

R.P. Boyer (1983). Infinite traces of AF-algebras and characters of $U(\infty)$, *J. Operator Theory*, **9**, 205-236.

Infinite symmetric group

E. Thoma (1964). Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbaar unendlichen, symmetrischen Gruppe, *Math. Z.*, **85**, 40–61.

A. Vershik, S. Kerov (1981). Asymptotic theory of characters of a symmetric group, *Funct. Anal. Appl.*, **15**, 246–255.

S. Kerov (2003). Asymptotic Representation Theory of the Symmetric Group and its Applications in Analysis. *Translations of Mathematical Monographs, A.M.S.*.

Some more recent papers

A. Okunkov, G. Olshanski (1998). Asymptotics of Jack polynomials as the number of variables goes to infinity, *Internat. Math. Res. Notices*, **13**, 641–682.

A. Okunkov, G. Olshanski (1997). Limits of BC-type orthogonal polynomials as the number of variables goes to infinity, *Contemporary Mathematics*, **-**, -.

G. Olshanski (2003). The problem of harmonic analysis on the infinite-dimensional unitary group, *J. Funct. Anal.*, **205**, 464–524.

A. Borodin, G. Olshanski (2005). Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes, *Annals of Math.*, **161**, 1319–1422.

T. Hirai, E. Hirai (2005). Characters of wreath products of finite groups with the infinite symmetric group, *J. Math. Kyoto Univ.*, **45**, 547–597.

M. Bouali (2006). Application des théorèmes de Minlos et Poincaré à l'étude asymptotique d'une intégrale orbitale, *Ann. Fac. Sci. Toulouse*, **14**, –.

M. Rabaoui (2007). Une généralisation du théorème de Bochner. *Submitted.*

Further reference

J. Faraut (2006). Infinite dimensional harmonic analysis and probability, in *Probability measures on groups: recent directions and trends,* (eds. S.G. Dani and P. Graczyk), Tata Institute of Fundamental Research, *Narosa, New Dehli*, 179–254.