Conformally Invariant Systems of Differential Operators

Joint with A. Kable and R. Zierau

1 Motivation

⋆ Kostant's Quasi-Invariant Differential Operators

Assume that σ is a representation of a Lie algebra $\mathfrak{g}$ by first order smooth differential operators on a manifold M. If Ω is a diff. operator so that

$$[\sigma(Y), \Omega] = 0, \text{ for all } Y \in \mathfrak{g},$$

then the solution space of Ω is $\mathfrak{g}$-invariant.

A weaker condition of Ω still guarantees an invariant solution space. A differential operator action on a manifold M is said to be *quasi-invariant* if there exists a smooth function $C^Y(\cdot)$ on M so that

$$[\sigma(Y), \Omega] = C^Y(\cdot)\Omega, \text{ for all } Y \in \mathfrak{g}.$$

A typical example of a *quasi-inavariant operator* is the Wave operator on Minkowski space.
This fact was observe by Seagal. The solution space is conformally invariant.

- The multiplier action on Minkowski space comes from the action of $(\pi_s, \mathfrak{g} = \mathfrak{so}(4,2))$ in $\mathrm{ind}_{\overline{P}}^{SO(4,2)}(\chi_{-s})$. Here the Lie algebra of $\overline{P}$ is $\overline{\mathfrak{p}} = \mathfrak{l} \oplus \mathfrak{n}$ and $\mathfrak{l} = \mathfrak{so}(3,1) + \mathbf{R}H_o$. The element H_o induces a grading of $\mathfrak{g}$ by eigenspaces,

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1, \text{with } \mathfrak{g}_0 = \mathfrak{l}, \mathfrak{g}_1 = \mathfrak{n} \equiv T(M^4).$$

For a particular s, the Wave operator

$$\Box == \sum_{i=1}^{3} \frac{\partial^2}{\partial^2 x_i} - \frac{\partial^2}{\partial^2 x_4},$$

is quasi-invariant under π_s.

- The action of $\mathfrak{g}$ on $C^\infty(G/\bar{P}, \chi_{-s})$, $s \in \mathbf{R}$, induces an action $(\pi_s, \mathfrak{g}, C^\infty(N))$.

$$\pi_s : \mathfrak{g} \longrightarrow \mathbf{D}(N)$$
$$(\pi_s(Y)\phi)(n) = s\chi(\mathrm{Ad}(n^{-1})Y)\,\phi(n) - R(\mathrm{Ad}(n^{-1}Y)\,\phi)(n).$$

- **Definition:** A differential operator D is quasi-invariant with respect to π_s if for each $y \in \mathfrak{g}$, there is a smooth function $C^Y(\cdot) \in C^\infty(N)$ such that

$$[\pi_s(Y), D] = C^Y(\cdot)\, D$$

A quasi-invariant operator D is *straight* if $C^Y(e) = 0$ for all $Y \in \mathfrak{n}$.

Proposition 1 *(Kostant) Every quasi-invariant differential operator is equivalent to a straight quasi-invariant operator.*

Theorem 2 *(Kostant) There is a bijections between all straight quasi-invariant differential operators and all leading weight vectors in the Generalized Verma module* $V_s = \mathcal{U}(\mathfrak{g}) \otimes_{\overline{\mathfrak{p}}} \boldsymbol{C}_s$

Remark 3 *If* V_s *is irreducible, then no quasi-invariant differential operator exists.*

$\star$Generalizations of Kostant Theory

$\bullet$ Intertwining differential operators between induced representations. Among the many contributions: Boe-Collingwood, J-S. Huang in connection to homomorphism of Generalized Verma modules and leading weight spaces. Korányi and Reimann, Orsted among those contributions describing explicitly intertwinig diff. op.

$\bullet$ Symmetries of differential operators.(Eastwood) Consider the Laplacian Δ on $\mathbf{R}^n$ or on a Riemannian manifold (the signature of the metric is irrelevant) a symmetry of Δ is a linear differential operator $\mathcal{D}$ so that

$$\Delta \mathcal{D} = \delta \Delta$$

for some linear differential operator δ. Two symmetries $\mathcal{D}_1$, $\mathcal{D}_2$ are equivalent if $\mathcal{D}_1 - \mathcal{D}_2 = \mathcal{P}\Delta$ for some linear diff. op. $\mathcal{P}$. The space of symmetris mod equivalence is closed under composition and determines an algebra $\mathcal{A}$. The problem is to describe $\mathcal{A}$.

Note that Δ quasi-invariant for $\mathfrak{g}$ under π_s, implies that

$$\Delta\pi_s(Y) = [\pi_s(Y) - C^Y]\Delta \text{ for all } Y \in \mathfrak{g}.$$

Hence $\mathcal{U}(\mathfrak{g})$ maps into the symmetry algebra of the operator Δ.

- Conformally invariant systems of differential operators.

1. When $M = G/\overline{P}$, there is a bijection between homogeneous conformally invariant systems of diff. op. on $G \times_{\overline{P}} E$ and leading weight spaces of $\mathcal{U}(\mathfrak{g}) \otimes_{\overline{\mathfrak{p}}} E^*$.
2. The theory of c.i.s is formulated for vector bundles over manifold other that homogenoeous spaces G/P.
3. The common kernel of the diff. operators in the system is $\mathfrak{g}$-invariant. One might hope to obtain interesting rep. of $\mathfrak{g}$ or even G on such spaces of smooth sections. This is the case in some examples.
4. A conjecture by Gyoja relates zeros of the Berstein-Sato polynomial to reducibility of Verma modules. When P is Heisenberg type we build c.i.s that produces the first reducibility point in each arithmetic progression predicted by Gyoja.
5. In examples, explicit constructions of c.i.s. lead to explicit singular homomorphism between generalized Verma modules.

Conformally Invariant Systems

Suppose that $\mathfrak{g}$ is a real or complex Lie algebra and M is a smooth manifold. To discuss invariance we require additional structure on M. Informally, we want

- to realize $\mathfrak{g}$ as an algebra of first order differential operators on M, in such a way that
- $\mathfrak{n} \subset \mathfrak{g}$ is identified with the tangent space of M at every point.

Assume that there is a map

$$\pi : \mathfrak{g} \to \mathbb{D}(M)$$

satisfying:
(1)$\pi([X,Y]) = [\pi(X), \pi(Y)]$.
(2) For each $X \in \mathfrak{g}$, $\pi(X)$ is a first order differential operator. $\pi(X) = \pi_0(X) + \pi_1(X)$, with $\pi_0(X)$ multiplication by a smooth function on M and $\pi_1(X)$ a vector field on M.
(3)For all $p \in M$, $X \to \pi(X)(p)$ is a linear iso between $\mathfrak{n}$ and $T_p(M)$.

A vector bundle $\mathcal{E} \to M$ is a $\mathfrak{g}$-bundle if there is a linear map

$$\pi_{\mathcal{E}} : \mathfrak{g} \to \mathbb{D}(\mathcal{E}), \text{such that}$$

$\pi_{\mathcal{E}}([X,Y]) = [\pi_{\mathcal{E}}(X), \pi_{\mathcal{E}}(Y)]$, and

$[\pi_{\mathcal{E}}(X), f] = \pi_1(X)f$,. for all $X, Y \in \mathfrak{g}$ and all $f \in C^\infty(M)$.

Definition 4 *A* conformally invariant system *on $\mathcal{E}$ is a set of differential operators*

$$D_1, \dots, D_n \in \mathbb{D}(\mathcal{E})$$

so that

1. *$\{D_1, \dots, D_n\}$ is linearly independent at each point of M, and*
2. *for each $X \in \mathfrak{g}$ there is an $n \times n$ matrix $C(X)$ of smooth functions on M*

$$[\pi_{\mathcal{E}}(X), D_j] = \sum_i C_{ij}(X) D_i.$$

•The map $C : \mathfrak{g} \to M_{n \times n}(C^\infty(M))$ is called the *structure operator.*

•A conformally invariant system is called *straight* if its structure operator vanishes on $\mathfrak{n}$.

•Two conformally invariant systems $D_1, \dots, D_n$ and $D'_1, \dots, D'_n$ are said to be equivalent if there is an invertible matrix in $A \in M_{n \times n}(C^\infty(M))$ so that

$$D'_j = \sum_i A_{ij} D_i.$$

Proposition 5 *Let M be simply-connected, and $\mathcal{E} \to M$ be a $\mathfrak{g}$-bundle. Then every conformally invariant system is equivalent to a straight conformally invariant system.*

Why is Proposition 5 true?

- If $\{D_1, \dots D_n\}$ is a c.i.s. on $\mathcal{E} \to M$, the trivial bundle over M spanned by $D_i's$ admits a $\mathfrak{g}$-bundle structure given by

$$\Pi_{\mathcal{E}(D)}(X) \cdot D_i = \sum_1^n C(X)_{i,j} D_j.$$

- Let $X_1, \dots, X_n$ be a basis of $\mathfrak{n}$ and $V_j = \pi(X_j)$. If $\mathcal{E} \to M$ is a $\mathfrak{g}$-bundle, there is a unique connection ∇ so that $\nabla_{V_j}(\sigma) = \pi_{\mathcal{E}}(X_j) \cdot \sigma$ for $\sigma \in \Gamma(\mathcal{E})$ and $1 \leq j \leq n$. This connection is flat.

- The holonomy group of ∇ is trivial. It follows that parallel transportation taking $\{D_i's\}$ as initial data produces sections $\{D_i'\}$ with $\Pi_{\mathcal{E}(D)}(X) \cdot D_i' = 0$.

⋆The Homogeneous Case

Assume that G is a real reductive Lie group, $P = LN$, $\overline{P} = L\overline{N}$. Given (σ, E) an irreducible finite dimensional rep. pf $\overline{P}$, form $C^\infty(G/\overline{P}, \mathcal{E})$ with G acting by left translation. Restriction to N gives an injection

$$C^\infty(G/\overline{P}, \mathcal{E}) \to C^\infty(N, E)$$

The Lie algebra action on the induced rep. induces an action on the image of the restriction map

$$(\pi_\sigma(Y)f)(n) = \sigma((\mathrm{Ad}(n^{-1})Y)_{\overline{\mathfrak{p}}})f(n) - (R((\mathrm{Ad}(n^{-1})Y)_{\mathfrak{n}})f)(n),$$

where $n \in N$. This action extend to an action on all $C^\infty(N, E)$ and $\mathcal{E} \to N$ is a $\mathfrak{g}$-bundle under π_σ.

Proposition 6 *Suppose that* $D_1, \ldots, D_n \in \mathbb{D}(\mathcal{E})$ *and each* D_j *commutes with* $\pi_\sigma(X)$, $X \in \mathfrak{n}$. *Assume that* $D_1, \ldots, D_n$ *are linearly independent at* e *and there is a map* $b : \mathfrak{g} \to \mathfrak{gl}(n, \boldsymbol{C})$ *so that*

$$([\pi_\sigma(X), D_j]f)(e) = \sum_{i=1}^{n} (b(X))_{ij}(D_i f)(e),$$

for all $X \in \mathfrak{g}$ *and all* $f \in C^\infty(N, E)$. *Then* $D_1, \ldots, D_n$ *is a straight conformally invariant system. The structure operator is given by* $C(X)(n) = b(Ad(n^{-1})X)$, *for* $n \in N$ *and* $X \in \mathfrak{g}$.

•Define

$$\mathcal{D}'_e(N, \mathcal{E}^*) \equiv \{\Lambda : C^\infty(N, \mathcal{E}) \to \mathbf{C} \ : \ \Lambda \text{ is continuous} \\ \text{and } \Lambda(f) = 0 \text{ if } e \notin \text{ supp}(f)\}.$$

Note that $\mathcal{D}'_e(N, \mathcal{E}^*)$ a $\mathcal{U}(\mathfrak{g})$-module by

$$(u \cdot \Lambda)(f) = \Lambda(\pi_\sigma(u^\circ)f), u \in \mathcal{U}(\mathfrak{g}),$$

where $u \longrightarrow u^\circ$ is the canonical anti-automorphism of $\mathcal{U}(\mathfrak{g})$.

• For $D \in \mathbb{D}(\mathcal{E})$ define

$$(D\Lambda_{e^*})(f) = \langle e^*, (Df)(e) \rangle.$$

If $\{D_1, \ldots D_n\}$ is a c.i.s.s and $Y \in \overline{\mathfrak{p}}$

$$Y \cdot (D_j \Lambda_{e^*}) = \sum_i C_{ij}(Y)(e) D_i \Lambda_{e^*} - D_j \Lambda_{\sigma^*(Y)e^*}.$$

Proposition 7 *If $D_1, \ldots, D_m$ is a conformally invariant system acting on $C^\infty(N, \mathcal{E})$, then*

$$F \equiv span\{D_j \Lambda_{e^*} \, : \, e^* \in E^*, j = 1, \ldots, m\}$$

is a $\overline{\mathfrak{p}}$-submodule of $\mathcal{D}'_e(N, \mathcal{E}^*)$.

Observations

- $\mathrm{Hom}_{\overline{\mathfrak{p}}}(F, \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\overline{\mathfrak{p}})} E^*) \neq \{0\}$.

- A simple condition on the c.i.s will guarantee that $F \subset \{\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\overline{\mathfrak{p}})} E^*\}^{\overline{\mathfrak{n}}}$, generalizing Kostant's Theorem.

Given $\mathfrak{p}$, fix $H_o \in \mathfrak{l}$ so that its eigenvalues are integers, $\mathfrak{g}(0) = \mathfrak{l}$ and $\mathfrak{n} = \oplus_{j>0} \mathfrak{g}(j)$.

A c.i.s. $\{D_1, \ldots D_n\}$ is called *homogeneous* if $C(H_o)$ is a scalar matrix.

Lemma 8 *If $D_1, \ldots, D_m$ is a homogeneous conformally invariant system with structure operator C, then $C(Y)(e) = 0$ for all $Y \in \overline{\mathfrak{n}}$.*

Theorem 9 *If $D_1, \ldots, D_m$ is a homogeneous c.i.s. acting on $C^\infty(N, \mathcal{E})$, then for the $\overline{\mathfrak{p}}$-rep.*
$F = span_{\mathbb{C}}\{D_j \Lambda_{e^*} \, : \, j = 1, \ldots, m, e^* \in E^*\}$,

$$Hom_{\mathfrak{l}}(F, \{\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\overline{\mathfrak{p}})} E^*\}^{\overline{\mathfrak{n}}}) \neq \{0\}.$$

Corollary 10 *If F is $\overline{P}$-stable, then* $\mathbb{D}_G(\mathcal{E},\mathcal{F}^*) \neq \{0\}$.

Claim: $F \subset \{\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\overline{\mathfrak{p}})} E^*\}^{\overline{\mathfrak{n}}}$, determines a *straight homogeneous c.i.s.*

- $(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\overline{\mathfrak{p}})} E^*) \otimes E \simeq \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\overline{\mathfrak{p}})} End(E) \simeq \mathbb{D}_N(\mathcal{E})$. The iso is induced by the map

$u \otimes T \mapsto D_{u\otimes T}$, where

$$\big(D_{u\otimes T} f\big)(n) = T\big((\pi_\sigma(u^\circ)(\ell_{n^{-1}} f))(e)\big).$$

- Fix a basis $f_1, \dots f_n \in F$ and $e_1, \dots e_m$ of E.

$f_j \otimes e_k \in \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\overline{\mathfrak{p}})} End(E)$.

Set $D_{jk} \equiv D_{f_j \otimes e_k} \in \mathbb{D}_N(\mathcal{E})$.

Write, for $Y \in \overline{p}$

$$Y \cdot f_j = \sum_i a_{ij}(Y) f_i,\ a_{ij}(Y) \in \mathbf{C},$$

$$\sigma(Y) e_k = \sum_l b_{lk}(Y) f_l,\ b_{lk}(Y) \in \mathbf{C}.$$

Then,

$$\big([\pi_\sigma(Y), D_{jk}] f\big)(e) = \sum_i a_{ij}(Y)(D_{ik} f)(e) + \sum_l b_{lk}(Y)(D_{jl} f)(e).$$

Theorem 11 *Suppose $F \subset \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\overline{\mathfrak{p}})} E^*$ is a $\overline{\mathfrak{p}}$-submodule and let f_j, e_k and D_{jk} be as above. Then $\{D_{jk}\}$ is a conformally invariant system.*

$\star$ **Conformally Invariant Systems for Parabolics with Abelian Nilradical**

- $P = LN$ with $\mathfrak{n}$ abelian.

- Schmid's Theorem gives the decomposition of $S(\mathfrak{n}) = \mathcal{U}(\mathfrak{n})$ as an L-module. All rep. occur with multiplicity one.

- If $\tilde{\mathfrak{n}}$ is the nilradical of a Borel in $\mathfrak{l}$, then $S(\mathfrak{n})^{\tilde{\mathfrak{n}}} = \mathbf{C}[u_1, \ldots, u_r]$ with $u_1^{m_1} \ldots u_r^{m_r}$ the highest weight vectors of the irreducible constituents in $S(\mathfrak{n})$.

- Let F_j be the irreducible rep. with highest weight vector u_j. A computation done by Wallach shows that there is a $s_j \in \mathbf{R}$ so that $F_j \otimes 1 \subset \mathcal{U}(\mathfrak{n}) \otimes \mathbf{C}_{s_j}$ is a leading weight space.

- A basis of polynomials in F_j, leads to differential operators on $C^\infty(N, \mathbf{C}_{-s_j})$ that are conformally invariant.

- If $G = GL(n, \mathbf{R})$, $L = GL(p, \mathbf{R}) \times GL(q, \mathbf{R})$, $\mathfrak{n} \equiv M_{p\times q})$,

$$F_j = \text{span}\{\det(j \times j \text{ minors})\}.$$

The construction of the conformally invariant system from F_j is as follows.
$\mathcal{P}(\overline{\mathfrak{n}}) \equiv S(\mathfrak{n})$ is iso to and the constant coefficient differential operators on $\mathfrak{n}$. Given $P(Y) \in \mathcal{P}(\overline{\mathfrak{n}})$, $P(\partial_x)$ is the constant coefficient differential operator satisfying

$$P(\partial_x)\, e^{\langle Y,X\rangle} = P(Y)\, e^{\langle Y,X\rangle}, \tag{12}$$

where $\langle Y, X\rangle = Tr(\begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix} \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}) = Tr(YX)$.

$\star$ Conformally Invariant Systems for Heisenberg Type Parabolics

- $\mathfrak{g}$ is a complex simple Lie algebra with rank $\mathfrak{g} > 1$.

- Choose $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra; $R = \Delta(\mathfrak{g}, \mathfrak{h})$, $R^+ = \Delta^+(\mathfrak{g}, \mathfrak{h})$.

$\gamma =$ highest root in R, $\{E_\gamma, E_{-\gamma}, H_\gamma\}$ is an $\mathfrak{s}l(2)$-triple.

- $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$, $\Delta(\mathfrak{p}) = \{\alpha \in R : \alpha(H_\gamma) \geq 0\}$.

- $\mathfrak{n} = V^+ \oplus \mathbf{C}E_\gamma$, $\mathbf{C}E_\gamma = z(\mathfrak{n})$.

- $G = \mathrm{Aut}(\mathfrak{g})^o$, $L \subset G$ connected with Lie algebra $\mathfrak{l}$.

- (L, Ad, V^+) is a prehomogeneous vector space.
- $\overline{\mathfrak{n}} =$ nilradical of $\overline{p}$ (opposite to $\mathfrak{p}$). $\overline{\mathfrak{n}} = V^- + \mathbf{C}E_{-\gamma}$.

- We have a $ad(H_\gamma)$-weight space decomposition:

$\mathfrak{g} = \mathfrak{g}_{-\gamma} \oplus V^- \oplus \mathfrak{l} \oplus V^+ \oplus \mathfrak{g}_\gamma$.

- $\mathfrak{g} = \mathrm{span}\{\mathfrak{n}, \mathfrak{l}, E_{-\gamma}\}$.
- Since $\mathfrak{g}_\gamma$ is one dimensional, there is a character $\chi : \mathfrak{l} \to \mathbf{C}^*$, so that

$$Y \in \mathfrak{g}_\gamma,\ \mathrm{Ad}\ (\ell)Y = \chi(\ell)^{-1}Y.$$

Let G_0 be a real simple Lie group with Lie algebra $\mathfrak{g}_0$ that parabolic subalgebra $\mathfrak{p}_0$ with $\mathfrak{p} = \mathfrak{p}_0 \otimes \mathbf{C}$ is a subalgebra of $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbf{C}$ of Heisenberg type.

Goal: Use Invariant theory to construct c.i.s on $C^\infty(G_0/\bar{P}_0, \chi_{-s})$.

The invariant theory

A *covariant* is :

an irreducible representation (ρ, W) of L along with a nonzero L-equivariant polynomial map

$$F : V^+ \to W.$$

- By a polynomial map mean a map for which each coordinate is polynomial in V^+.

- a covariant comes from an element of $\{P(V^+) \otimes W\}^L$. But this is $\mathrm{Hom}_L(W^*, P(V^+))$.

- There are four natural covariants $\tau_1, \ldots, \tau_4$ which will play a role in our construction. For $X \in V^+$, $\tau_k(X) \in \mathfrak{g}$ is given by

$$\tau_k(X) = \frac{1}{k!}\mathrm{ad}(X)^k(E_{-\gamma}).$$

$$\tau_4(X) = \Delta(X) E_\gamma$$

Lemma 13 *For $\ell \in L$, $X \in V^+$ and $1 \leq k \leq 4$, we have*

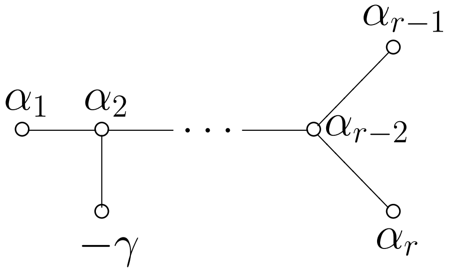
$$\tau_k(Ad(\ell)X) = \chi(\ell)\, Ad(\ell)\, \tau_k(X).$$

Facts:

- If $\mathfrak{g}$ is not of type C_r and prove that $\tau_k \neq 0$ for $1 \leq k \leq 4$. Under this assumption there exists a simple root $\delta \in \Delta(V^+, \mathfrak{h})$ so that $\delta' = \gamma - \delta \in \Delta(V^+, \mathfrak{h})$ and $\gamma - 2\delta$ is not a root.
For non-zero scalars x, y

$$\tau_k(xE_\delta + yE_{\gamma-\delta}) \neq 0.$$

Example 14 *If* $\mathfrak{g} = \mathfrak{so}(2r, \boldsymbol{C})$*, the extended Dynkin diagram is*



$\delta = \alpha_2$.
(Ad, V^+) *is equivalent to* $GL(2) \times SO(2r-4)$ *acting on the tensor product* $\boldsymbol{C}^2 \otimes \boldsymbol{C}^{2r-4}$.

- The point $X_o = \sqrt{2}\big(E_\delta + E_{\delta'}\big) \in V^+$ is a generic point of the prehomogeneous vector space (ad, V^+).

- Since $\mathfrak{n} = V^+ + \mathfrak{z}$, V^+ admits a non-degenrate alternating form ω given by

$$[X_1, X_2] = \omega(X_1, X_2)E_\gamma, \text{ for } X_1, X_2 \in V^+.$$

By applying $\mathrm{Ad}(\ell)$ to the above formula we have

$$\omega(\mathrm{Ad}(\ell)X_1, \mathrm{Ad}(\ell)X_2) = \chi(\ell)\omega(X_1, X_2).$$

Plan:

(1) The covariants $\tau_1, \tau_2, \tau_3, \tau_4$ will be used to find $\mathfrak{l}$-subrepresentations of $\mathcal{U}(\mathfrak{n})$. We have

$$\tau_k : V^+ \to W_k,\ \ k = 1, \ldots, 4$$

are L-equivariant polynomial maps. That is

$$\tau_k \in \{P(V^+) \otimes W_k\}^L \simeq \mathrm{Hom}_L(W_k^*, P(V^+)).$$

The symplectic form ω gives an isomorphism

$$(V^+)^* \simeq V^+ \otimes \mathbf{C}_{\chi^{-1}}.$$

Now

$$P(V^+) \simeq S((V^+)^*) \simeq S(V^+ \otimes \mathbf{C}_{\chi^{-1}})$$

. Since W_k occurs as homogeneous polynomials of degree k, we have

$$W_k^* \to S^k(V^+) \otimes \mathbf{C}_{\chi^{-k}},$$

therefore

$$W_k^* \otimes \mathbf{C}_{\chi^k} \to S^k(V^+) \hookrightarrow \mathcal{U}(\mathfrak{n}).$$

(2) Extend χ^s, $s \in \mathbf{R}$, of L to a rep. of $\bar{P}$ by making $\bar{N}$ act trivially.
The differential of χ^s has weight $s\gamma$. Then consider $C^\infty(N, \mathbf{C}_{\chi^{-s}})$ and the action of $\mathfrak{g}$ by π_s.
Each embedding

$$W_k^* \otimes \mathbf{C}_{\chi^k} \to S(V^+) \hookrightarrow \mathcal{U}(\mathfrak{n})$$

gives a family of differential operators by the right action.

(3) Determine the values of s for which the operators are conformally invariant with respect to π_s.

k	$W_k^* \otimes \mathbf{C}_{\chi^k}$
1	V^+
2	$[\mathfrak{l}, \mathfrak{l}] \otimes \mathbf{C}_\chi$
3	$V^- \otimes \mathbf{C}_{\chi^2}$
4	$\mathbf{C}_{\chi^2}$

- When $k = 1$ case, the embedding of V^+ into $S(V^+)$ is clear.

- When $k = 2$, τ_2 gives $[\mathfrak{l}, \mathfrak{l}] \otimes \mathbf{C}_\chi \subset P(V^+)$.

Each $Z \in [\mathfrak{l}, \mathfrak{l}]$ maps to the polynomial (in X)

$$\kappa(\tau_2(X), Z).$$

After identifying $P(V^+)$ with $S(V^+)$ we obtain the map $[\mathfrak{l}, \mathfrak{l}] \otimes \mathbf{C}_\chi \to S(V^+)$ given by

$$Z \mapsto \frac{1}{2} \sum_{\alpha, \beta \in \Delta(V^+)} \omega_\beta^{-1} M_{\alpha, \beta'}(Z)(E_\alpha E_\beta + E_\beta E_\alpha),$$

where

$$\mathrm{ad}(Z)(E_\alpha) = \sum_{\beta \in \Delta(V^+)} M_{\beta, \alpha}(Z) E_\beta.$$

- Similar computations may be done for τ_3 and τ_4.

$\star$ When $k = 1$, the embedding $V^+ \hookrightarrow \mathcal{U}(\mathfrak{n})$ gives a family of differential operators

$$\Omega_1 = \{R(E_\alpha) \,:\, \alpha \in \Delta(V^+)\}$$

Proposition 15 *When $s = 0$, Ω_1 is a conformally invariant systems.*

$\star$ When $k = 2$, we need some notation:

- Let $D(\mathfrak{g}, \mathfrak{h})$ be the Dynkin graph of $\mathfrak{g}$ with respect to $\mathfrak{h}$. $D_\gamma(\mathfrak{g}, \mathfrak{h})$ the subgraph obtained by deleting the nodes attached to $-\gamma$ in the extended Dynkin diagram.

- $\mathcal{C}$ = a component of $D_\gamma(\mathfrak{g}, \mathfrak{h})$. ($D_\gamma(\mathfrak{g}, \mathfrak{h})$ is connected except in types B_r and D_r). $R(\mathfrak{l}, \mathcal{C}) = \text{span}\ \{\alpha \in R(\mathfrak{l}),\ \text{simple in}\ \mathcal{C}\}$.

- $\ell(\mathcal{C}) = \{X_\lambda : \lambda \in R(\mathfrak{l}, \mathcal{C})\}$..

Theorem 16 *Let $\mathcal{C}$ be a component of $D_\gamma(\mathfrak{g}, \mathfrak{h})$ and $Z \in \ell(\mathcal{C})$.*

$$[\pi_s(X_{-\gamma}), \Omega_2(Z)] = \\ -2\,\xi_\gamma \Omega_2(Z) + \Omega_2([\tau_2(X), Z]) + (s - s_2)\,\Omega_1([Z, X]),$$

where $s_2 = (1/2)\, c(\mathfrak{g}, \mathcal{C}) - 1$.

Theorem 17 *In type A_r,*

$$[\pi_s(X_{-\gamma}), \Omega_2(Z_o)] = \\ -2\,\xi_\gamma \Omega_2(Z_o) + s - \frac{r-1}{2}\,\Omega_1([Z_o, X]).$$

- There are two copies of V^- inside $S(\mathfrak{n})$. One such copy is obtained via τ_3 leading to a system of differential operators $\Omega'_3(Y)$. The second copy produces operators of the form $C_3(Y) = \Omega_1([Y, E_\gamma])\partial_\gamma$ for each $Y \in V^-$. It is a linear combination of these operators that produces a system $\{\Omega_3(Y)\}$ that is c.i. for appropriate s.

Theorem 18 *If* $\mathfrak{g}$ *is an exceptional algebra, then* $\{\Omega_3(Y), Y \in V^-\}$ *is conformally invariant iff* $s = (dim\ (V^+) - 2)/6$.

Theorem 19 *The system* Ω_3 *exists for the simple algebras of type* A_2 *and* D_4*. It does not exist for the simple algebras of types* A_r *with* $r \geq 3$*,* B_r *with* $r \geq 3$*, nor* D_r *with* $r \geq 5$.

$\star$ **Remark:**For $X + yE_\gamma \in V^+ \oplus \mathfrak{z}$, consider the polynomial

$$P(X + yX_\gamma) = y^2 - \Delta(X),\ X \in V^+,\ y \in \mathbf{C}.$$

Gyoja conjectured that the Generalized Verma module $\mathcal{U}(\mathfrak{g}) \otimes_{\overline{\mathfrak{p}}} \mathbf{C}_s$ is reducible iff $b(-s-k) = 0$ for $k \geq 1$. Our construction produces the first reducibility point in each arithmetic progression predicted by Gyoja.

- Example :B_r, $r \geq 3$. .

The b-function of the quasi-invariant is conjectured to be

$$b(s) = (s+1)\,(s+2)\,(s + (r - \frac{3}{2}))\,(s + (r-1)).$$

Type	Ω_1	Ω_2^{big}	Ω_2^{small}	Ω_3	Ω_4
$A_r (r \geq 2)$	0	0	$\frac{r-1}{2}$	none	$\frac{r-2}{2}$
$B_r (r \geq 3)$	0	$r - \frac{5}{2}$	1	none	$r-2$
$C_r (r \geq 2)$	0	$\frac{-1}{2}$	none	none	none
$D_r (r \geq 5)$	0	$r-3$	1	none	$r - \frac{5}{2}$
E_6	0	2	–	3	$\frac{9}{2}$
E_7	0	3	–	5	$\frac{15}{2}$
E_8	0	5	–	9	$\frac{27}{2}$
F_4	0	$\frac{3}{2}$	–	2	3
G_2	0	$\frac{2}{3}$	–	$\frac{1}{3}$	$\frac{1}{2}$

In type D_4 , $\mathcal{D}_\gamma(\mathfrak{g}, \mathfrak{h})$ has three components. The special s values for D_4 are

Type	Ω_1	$\Omega_{2(1)}$	$\Omega_{2(2)}$	$\Omega_{2(3)}$	Ω_3	Ω_4
D_4	0	1	1	1	1	$\frac{3}{2}$