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on the occasion of his 80th birthday
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**The Schrödinger equation
on symmetric spaces**

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1. Introduction

Schrödinger equation :

$$\begin{cases} i \partial_t u(t, x) + \Delta_x u(t, x) = g(t, x) \\ u(0, x) = f(x) \end{cases} \quad (1)$$

Wave equation :

$$\begin{cases} \partial_t^2 u(t, x) - \Delta_x u(t, x) = g(t, x) \\ u(0, x) = f_0(x), \quad \partial_t|_{t=0} u(t, x) = f_1(x) \end{cases} \quad (2)$$

Homogeneous equations: $g = 0 \rightsquigarrow$ linear

Typical nonlinearities :

$$g(t, x) \doteq \begin{cases} u(t, x) |u(t, x)|^\gamma \\ |u(t, x)|^{1+\gamma} \end{cases}$$

Setting : Riemannian manifold

$$\text{e.g. symmetric space} \begin{cases} \mathbb{R}^n \\ \mathbb{S}^n \text{ sphere} \\ \mathbb{H}^n \text{ hyperbolic space} \end{cases}$$

2. Schrödinger in \mathbb{R}^n

Solution to the Schrödinger equation (1):

$$u(t, x) = \underbrace{e^{it\Delta} f(x)}_{F(t, x)} + \underbrace{\int_0^t e^{i(t-s)\Delta} g(s, x) ds}_{G(t, x)}$$

Homogeneous solution

$$F(t, x) = e^{it\Delta} f(x) = (f * s_t)(x)$$

given by explicit kernel

$$s_t(x) = 2^{-n} \pi^{-\frac{n}{2}} e^{\text{sign}(t) i \frac{\pi}{4} n} |t|^{-\frac{n}{2}} e^{-\frac{i}{4} \frac{|x|^2}{t}}$$

(heat kernel with imaginary time)

Two main estimates:

- **Dispersive estimates:**

$$\|e^{it\Delta}\|_{L^1 \rightarrow L^\infty} = \|s_t\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}}$$

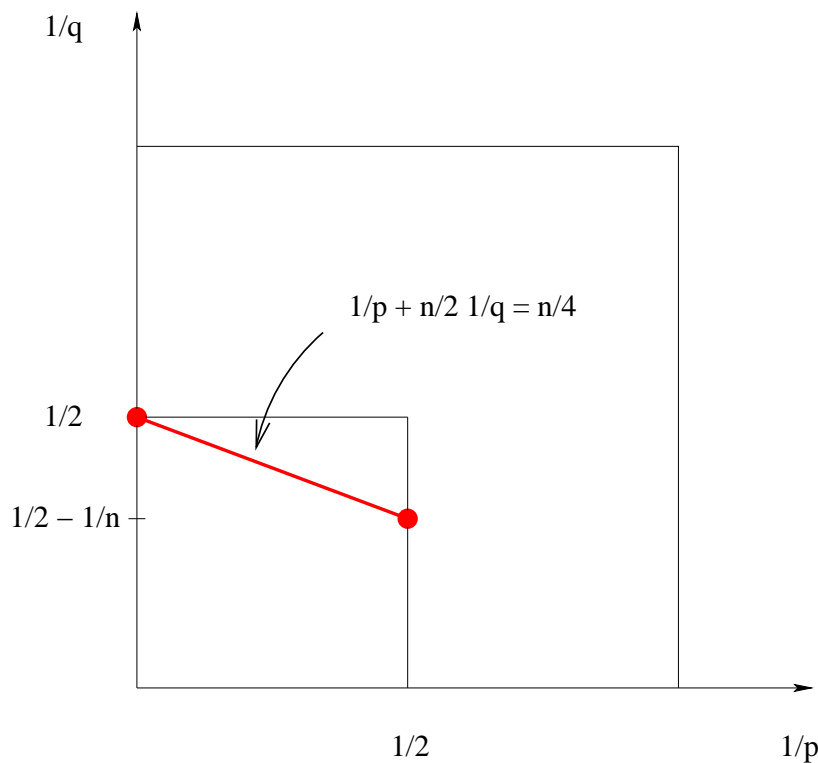
$$\Rightarrow \|e^{it\Delta}\|_{L^{q'} \rightarrow L^q} \lesssim |t|^{-n(\frac{1}{2} - \frac{1}{q})} \quad \forall 2 \leq q \leq \infty$$

$$\Rightarrow \|e^{it\Delta}\|_{L^{q'} \rightarrow L^{\tilde{q}}} \lesssim |t|^{-\frac{n}{2}(1 - \frac{1}{q} - \frac{1}{\tilde{q}})} \quad \forall 2 \leq q, \tilde{q} \leq \infty$$

- **Strichartz type estimates :**

$$\|u(t, x)\|_{L_t^{\tilde{p}} L_x^{\tilde{q}}} \lesssim \|f(x)\|_{L_x^2} + \|g(t, x)\|_{L_t^{p'} L_x^{q'}} \quad (3)$$

$\forall (p, q), (\tilde{p}, \tilde{q})$ in the *admissible* interval



In dimension $n > 2$, the estimate (3) holds true at the endpoint $(\frac{1}{2}, \frac{1}{2} - \frac{1}{n})$

References : [Ginibre–Velo], [Keel–Tao]

Wellposedness (existence and uniqueness)
for NLS (NonLinear Schrödinger equation) :

- $g(t, x) \doteq u(t, x) |u(t, x)|^\gamma \quad (\gamma > 0)$

conservation of $\begin{cases} L^2 \text{ mass} \\ H^1 \text{ energy} \end{cases}$

wellposedness	in $L^2(\mathbb{R}^n)$	in $H^1(\mathbb{R}^n)$
global	$\gamma < \frac{4}{n}$	$\gamma < \frac{4}{n-2}$
global	$\gamma = \frac{4}{n}$ s.i.d.	$\gamma = \frac{4}{n-2}$ s.i.d.

s.i.d. = small initial data

- $g(t, x) \doteq |u(t, x)|^{1+\gamma} \quad (\gamma > 0)$

no conservation !

wellposedness	in $L^2(\mathbb{R}^n)$	in $H^1(\mathbb{R}^n)$
local	$\gamma < \frac{4}{n}$	$\gamma < \frac{4}{n-2}$ s.i.d.
global	$\gamma = \frac{4}{n}$ s.i.d.	$\gamma = \frac{4}{n-2}$ s.i.d.

Reference : [Cazenave]

Scattering for NLS

Strichartz estimates

\rightsquigarrow global solution $u(t, x) \in L_t^p L_x^q$

$t \rightarrow \pm\infty : u(t, x) \rightarrow 0$ in $L_t^p L_x^q$

\rightsquigarrow NLS approaches HS

$\rightsquigarrow u(t, x)$ close to $e^{it\Delta} u_{\pm}(x)$

Scattering for NLS in $L^2(\mathbb{R}^n)$:

For any global solution $u \in C(\mathbb{R}, L^2(\mathbb{R}^n))$,

do there exist $u_{\pm} \in L^2(\mathbb{R}^n)$ such that

$$\|u(t, x) - e^{it\Delta} u_{\pm}(x)\|_{L_x^2} \rightarrow 0 \text{ as } t \rightarrow \pm\infty ?$$

Idem for H^1 scattering

• $g(t, x) \doteq u(t, x) |u(t, x)|^{\gamma} \quad (\gamma > 0)$

$$\begin{cases} L^2 \text{ scattering for } \gamma = \frac{4}{n} \text{ and s.i.d} \\ H^1 \text{ scattering for } \frac{4}{n} < \gamma < \frac{4}{n-2} \\ \text{no scattering for } 0 < \gamma < \frac{2}{n} \end{cases}$$

• $g(t, x) \doteq |u(t, x)|^{1+\gamma} \quad (\gamma > 0)$

no result (to our knowledge)

Reference : [Tao–Visan–Zhang]

3. NLS on manifolds

Aim : extend theory from Euclidean space to Riemannian manifold e.g. symmetric space

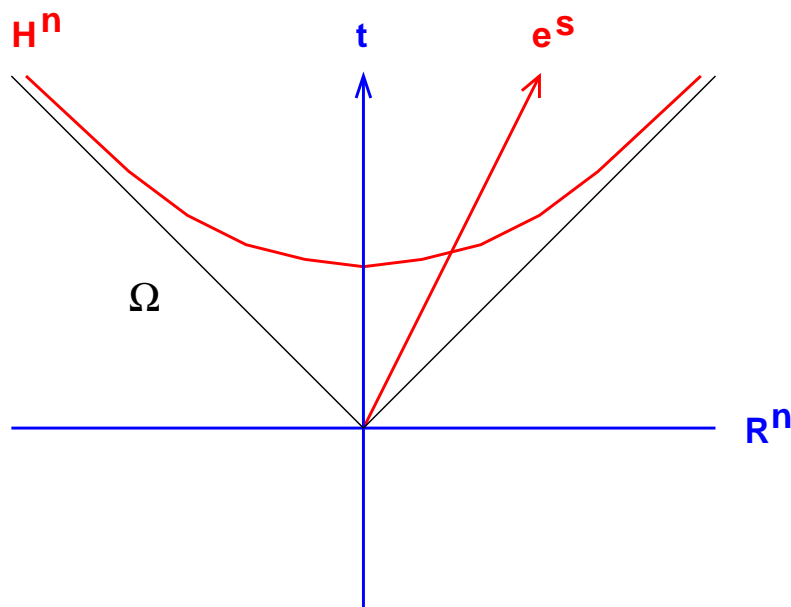
Motivations :

- understand influence of geometry
- compact case :
 - expect less dispersion \rightsquigarrow weaker results
- noncompact case :
 - expect more dispersion \rightsquigarrow stronger results
- transfer results (in both directions)
 - between the flat case and the curved case

Parenthesis :

- [Georgiev–Lindblat–Sogge]
 - GWP for NLW in \mathbb{R}^n
- [Tataru] shortcut via \mathbb{H}^n

$$\text{Light cone } \Omega \begin{cases} t^2 - x_1^2 - \dots - x_n^2 > 0 \\ t > 0 \end{cases}$$



$$\begin{aligned} \square &= \partial_t^2 - \Delta_{\mathbb{R}^n} \\ &= e^{-\frac{n+3}{2}s} \circ \underbrace{\left\{ \partial_s^2 - \Delta_{\mathbb{H}^n} - \left(\frac{n-1}{2}\right)^2 \right\}}_{-\mathcal{L}} \circ e^{\frac{n-1}{2}s} \end{aligned}$$

dispersive estimates on \mathbb{H}^n
with exponential decay

\rightsquigarrow GWP for NLW on \mathbb{R}^n

weighted Strichartz estimates on $\mathbb{H}^n \rightsquigarrow \mathbb{R}^n$

NLS on compact manifolds M

- [Burq–Gérard–Tzvetkov]
local Strichartz estimates
with loss of derivatives

$$\|e^{it\Delta} f(x)\|_{L^p(I;L^q(M))} \leq C_I \|f\|_{H^{1/p}(M)}$$

- [Bourgain]
LWP on $H^s(\mathbb{T}^n)$ when $s > \frac{n-2}{2}$

- [Burq–Gérard–Tzvetkov]
 $\left\{ \begin{array}{l} \text{LWP on } H^s(M) \text{ when } s > \frac{n-1}{2} \\ \text{LWP on } H^s(\mathbb{S}^n) \text{ when } s > \frac{n-2}{2} \end{array} \right.$

- [Gérard–Pierfelice]
GWP on $H^1(M)$
for 4–dimensional manifolds

NLS on hyperbolic space \mathbb{H}^n

Previous results :

- [Fontaine]
- [Tataru]
- [Banica] weighted dispersive estimate

$$\left\{ \begin{array}{l} \text{dimension } n \geq 3 \\ \text{radial functions} \\ \text{weight } w(x) = \frac{\sinh|x|}{|x|} \end{array} \right.$$

$$\|w(x)u(t, x)\|_{L_x^\infty} \lesssim \left\{ |t|^{-\frac{n}{2}} + |t|^{-\frac{3}{2}} \right\} \|w(x)^{-1}f(x)\|_{L_x^1}$$

- [Pierfelice] weighted Strichartz estimate

$$\left\{ \begin{array}{l} \text{Damek–Ricci spaces} \\ \text{radial functions} \\ \text{weight } w(x) = \left(\frac{\sinh|x|}{|x|}\right)^{m_1} \left(\frac{\sinh 2|x|}{2|x|}\right)^{m_2} \\ \text{same admissible interval as } \mathbb{R}^n \end{array} \right.$$

$$\|w(x)^{\frac{1}{2}-\frac{1}{q}}u(t, x)\|_{L_t^{\tilde{p}}L_x^{\tilde{q}}} \lesssim \|f(x)\|_{L_x^2} + \|w(x)^{\frac{1}{q}-\frac{1}{2}}g(t, x)\|_{L_t^{p'}L_x^{q'}}$$

- [Banica–Carles–Staffilani]
application to NLS and scattering
for radial functions

- [Chanillo]
partial results for G/K with G complex

Kernel expression : $r = |x|$

• n odd :

$$s_t(r) = 2^{-\frac{n+1}{2}} \pi^{-\frac{n}{2}} e^{i \operatorname{sign}(t) \frac{\pi}{4}} |t|^{-\frac{1}{2}} e^{i \left(\frac{n-1}{2}\right)^2 t} \\ \times \left(-\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^{\frac{n-1}{2}} e^{-\frac{i}{4} \frac{r^2}{t}}$$

• n even :

$$s_t(r) = 2^{-\frac{n+1}{2}} \pi^{-\frac{n+1}{2}} e^{i \operatorname{sign}(t) \frac{\pi}{4}} |t|^{-\frac{1}{2}} e^{i \left(\frac{n-1}{2}\right)^2 t} \\ \times \int_r^\infty ds \frac{\sinh s}{\sqrt{\cosh s - \cosh r}} \left(-\frac{1}{\sinh s} \frac{\partial}{\partial s} \right)^{\frac{n}{2}} e^{-\frac{i}{4} \frac{s^2}{t}}$$

Kernel estimates : $t \neq 0$

$$|s_t(r)| \lesssim \begin{cases} |t|^{-n/2} j(r)^{-1/2} & \text{if } |t| \leq 1+r \\ |t|^{-3/2} \varphi_0(r) & \text{if } |t| \geq 1+r \end{cases}$$

where $\begin{cases} j(r)^{-1/2} \asymp (1+r)^{\frac{n-1}{2}} e^{-\frac{n-1}{2}r} \\ \varphi_0(r) \asymp (1+r) e^{-\frac{n-1}{2}r} \end{cases}$

$$\Rightarrow \|s_t\|_{L^q} \lesssim \begin{cases} |t|^{-n/2} & \text{if } 0 < |t| \leq 1 \\ |t|^{-3/2} & \text{if } |t| \geq 1 \end{cases} \quad \forall q > 2$$

Idem for $\|s_t\|_{L^{q,a}}$ $\forall q > 2$ and $\forall a \geq 1$

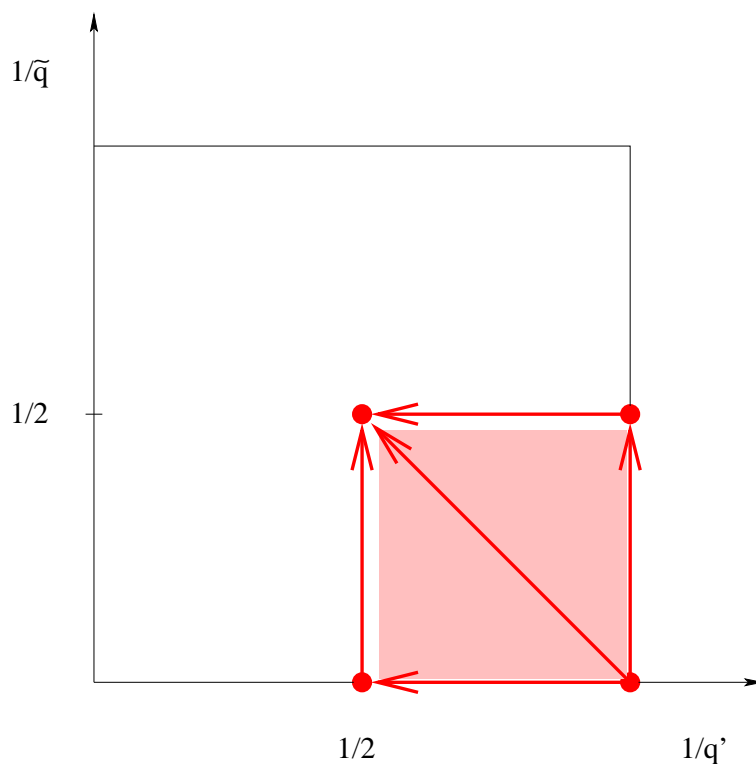
Dispersive estimates: $2 \leq q, \tilde{q} \leq \infty$

- $0 < |t| \leq 1$:

$$\|e^{it\Delta}\|_{L^{q'} \rightarrow L^{\tilde{q}}} \lesssim |t|^{-\frac{n}{2}(1-\frac{1}{q}-\frac{1}{\tilde{q}})}$$

- $|t| \geq 1$:

$$\|e^{it\Delta}\|_{L^{q'} \rightarrow L^{\tilde{q}}} \lesssim \begin{cases} 1 & \text{if } q = \tilde{q} = 2 \\ |t|^{-\frac{3}{2}} & \text{if } 2 < q, \tilde{q} \leq \infty \\ |t|^{-\frac{3}{4}} & \text{otherwise} \end{cases}$$



Proof: Interpolation

Crucial estimate: $\|e^{it\Delta}\|_{L^{q'} \rightarrow L^q} \lesssim |t|^{-\frac{3}{2}}$ $\begin{cases} |t| \geq 1 \\ q > 2 \end{cases}$

follows from

- $\|s_t\|_{L^q,1} \lesssim |t|^{-\frac{3}{2}}$

- the sharp Kunze–Stein phenomenon:

[Ionescu 2000] $L^{2,1}(G) * L^{2,1}(G) \subset L^{2,\infty}(G)$

\Rightarrow [Cowling 1997] $L^{q',a}(G) * L^{q',b}(G) \subset L^{q',c}(G)$

if $1 \leq q' < 2$ and $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} \geq 1$

$\Rightarrow L^{q'}(K \backslash G/K) * L^{q'}(G/K) \subset L^{q',c'}(K \backslash G/K)$

if $1 \leq q' < 2$ and $\frac{1}{c'} \leq \frac{2}{q'} - 1$

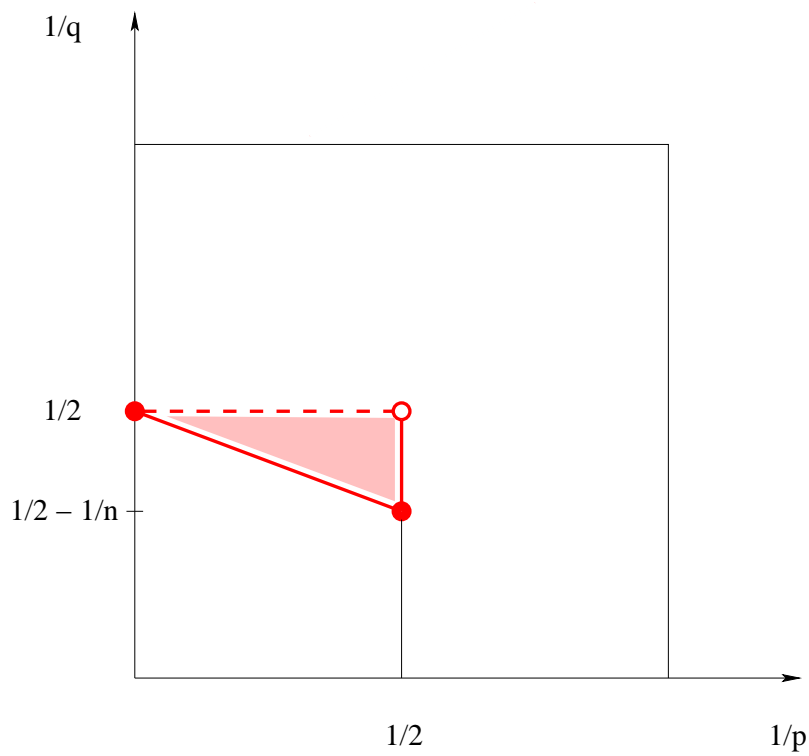
$\Rightarrow L^{q'}(G/K) * L^{q,c}(K \backslash G/K) \subset L^q(G/K)$

if $q > 2$ and $c \leq \frac{q}{2}$

Strichartz estimates :

$$\|u(t, x)\|_{L_t^{\tilde{p}} L_x^{\tilde{q}}} \lesssim \|f(x)\|_{L_x^2} + \|g(t, x)\|_{L_t^{p'} L_x^{q'}}$$

$\forall (p, q), (\tilde{p}, \tilde{q})$ in the *admissible* triangle



Wellposedness

Same results as in \mathbb{R}^n
with the following improvement

Theorem :

Assume $|g(t, x)| \lesssim |u(t, x)|^{1+\gamma}$ with $\gamma > 0$

Then global wellposedness for s.i.d.

$$\begin{cases} \text{in } L^2(\mathbb{H}^n) & \text{if } \gamma \leq \frac{4}{n} \\ \text{in } H^1(\mathbb{H}^n) & \text{if } \gamma \leq \frac{4}{n-2} \end{cases}$$

Scattering

Theorem : Same assumption. Then

$$\begin{cases} L^2 \text{ scattering for } \gamma \leq \frac{4}{n} \text{ and s.i.d.} \\ H^1 \text{ scattering for } \gamma \leq \frac{4}{n-2} \text{ and s.i.d.} \end{cases}$$

In contrast with \mathbb{R}^n , no long range effect
 \rightsquigarrow scattering for $\gamma > 0$ arbitrarily small

4. Further results and problems

NLW (NonLinear Wave equation)

Smoothing properties

NLS and NLW

- Damek–Ricci spaces
- higher rank noncompact symmetric spaces
- compact symmetric spaces
- discrete setting
 - trees, hyperbolic groups, ...
 - buildings
- special functions related to root systems
 - Dunkl (rational)
 - Heckman–Opdam (trigonometric)
 - Cherednik (q -theory)