Monodromy and Galois group for complex linear q-difference equations

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Goal : Galois theory by transcendental means (versus algebraic theory, see [André] or [van der Put & Singer]).

"Anybody who considers producing Galois groups *by* transcendental *means is, of course, in a state of sin."* John Von Neumann †

I Origins

II Fuchsian systems

III Local study of irregular systems

IV Geometric perspectives

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†If one replaces "Galois group" by "random numbers" and "transcendental" by "mechanical" ... .

I Origins

I.1 Generalities

Once and for all, $q \in \mathbf{C}$, $|q| > 1$. The field of coefficients:

$$\begin{aligned} K &= \mathbf{C}(z) \quad \text{(in the global case) or} \\ K &= \mathbf{C}(\{z\}) \text{ (in the local analytic case) or} \\ K &= \mathbf{C}((z)) \text{ (in the formal case),} \end{aligned}$$

is equipped with the automorphism $\sigma_q : f(z) \mapsto f(qz)$.

A q-difference equation:

$$\sigma_q^n f + a_1 \sigma_q^{n-1} f + \cdots + a_n f = 0 \ , \ a_i \in K \ , \ a_n \neq 0,$$

can be written $P(\sigma_q)(f) = 0$, $P \in \mathcal{D}_{q,K}$, where $\mathcal{D}_{q,K}$ is the Öre ring $K < \sigma, \sigma^{-1} >$ of q-difference operators.

Equations can be turned into systems, then into q-difference modules (see next slide); and vice versa.

A q-difference equation has a *Newton polygon at* 0, which consists in slopes $\mu_1 > \cdots > \mu_k \in \mathbf{Q}$ together with their multiplicities $r_1, \ldots, r_k \in \mathbf{N}^*$.

The Newton polygon depends only on the isomorphism class of the associated module .

I.2 The category $DiffMod(K, \sigma_q)$

The category of q-difference systems has:

1. Objects: systems $\sigma_q X = AX$, $A \in GL_n(K)$.

2. Morphisms $A \to B$: all $F \in M_{p,n}(K)$ such that $(\sigma_q F)A = BF$.

Morphisms are gauge transformations (they carry solutions X of $\sigma_q X = AX$ to solutions $Y = FX$ of $\sigma_q Y = BY$) and isomorphisms are gauge equivalences.

More intrinsically the category $DiffMod(K, \sigma_q)$ of q-difference modules over the q-difference field (K, σ_q) has:

1. Objects: pairs (V, Φ), where V is a finite dimensional K-vector space and Φ is a σ_q-linear automorphism of V (*i.e.* $\Phi(\lambda x) = \sigma_q(\lambda)\Phi(x)$).

2. Morphisms $(V, \Phi) \to (V', \Phi')$: all K-linear maps $u : V \to V'$ such that $\Phi' \circ u = u \circ \Phi$.

Equivalently, $DiffMod(K, \sigma_q)$ is the category of finite length left $\mathcal{D}_{q,K}$-modules. It is equipped with linear operations: tensor product, internal Hom, duals.

It is actually a tannakian category.

An equation is fuchsian at 0 if it is pure (only one slope) of slope 0. Then, for a fuchsian system, up to gauge equivalence, one can assume $A(0) \in GL_n(\mathbf{C})$: this is the definition used in [Birkhoff].

Fundamental Lemma. - Every fuchsian system is (locally) equivalent to one with coefficients in $\mathbf{C}$.

One gets explicit fundamental solutions from elementary functions s.t. $\sigma_q e_c = c e_c$ and $\sigma_q l = l + 1$ and convergent power series; classically, $e_c = z^{\log_q(c)}$ and $l = \log_q(z)$.

To a rational system $A(z)$ fuchsian at 0 and ∞, Birkhoff associates two local solutions $X^{(0)}$ and $X^{(\infty)}$ and their *connection matrix*:

$$P = \left(X^{(\infty)}\right)^{-1} X^{(0)}.$$

The latter is σ_q-invariant, hence (almost) elliptic.

He asks and solves (generically) the Generalized Riemann Problem: to recover $A(z)$ (up to rational equivalence) from P and local data at 0 and ∞ (exponents).

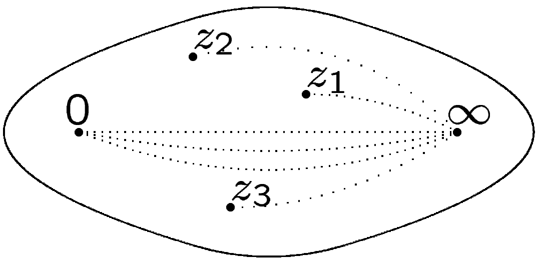
Problem: to interpret the connection matrix in galoisian terms or in terms of monodromy.
By the way: same problem for the local data.

II Fuchsian systems

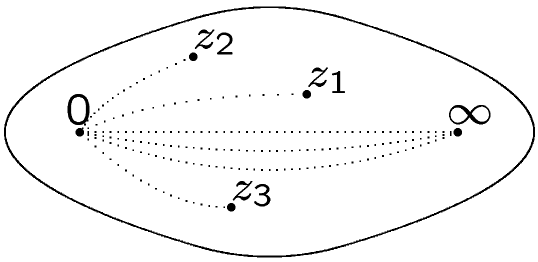
II.1 Single valued solutions

Following [Ramis], one builds *single valued* fundamental solutions in $\mathcal{M}(\mathbf{C}^*)$. This uses Jacobi's theta function: $\theta_q \in \mathcal{O}(\mathbf{C}^*)$ which satisfies $\sigma_q \theta_q = -qz\theta_q$.

Instead of ramification at 0 and ∞, the solutions inherit discrete logarithmic q-spirals (or half-spirals) of poles.



Singularities of $X^{(0)}$



Singularities of $X^{(\infty)}$

Now, the Birkhoff connection matrix P is truly elliptic: $P \in GL_n(\mathcal{M}(\mathbf{E}_q))$, where $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$.

Problem: its formation is not tensor compatible; moreover, this is unavoidable with meromorphic functions, because one must have $e_c e_d \neq e_{cd}$.

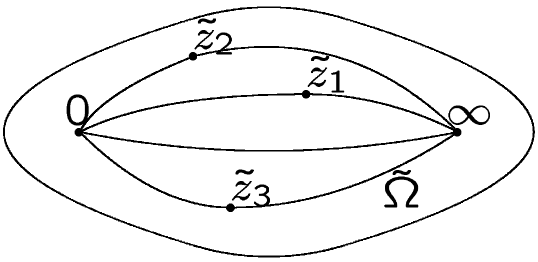
II.2 Confluency from qDE to ODE

[Sauloy 2000]

Classical q-analogy: $\delta_q = \frac{\sigma_q - 1}{q-1} \to \delta = z\frac{d}{dz}$ when $q \to 1$.

Suppose $(A_q - I_n)/(q-1) \to \tilde{B}$. Then, when $q \to 1$, under adequate conditions, a (single valued) fundamental solution $X_q^{(0)}$ of $\sigma_q X = A_q X$ over $\mathbf{C}^*$ converges to a (multivalued) fundamental solution $\tilde{X}^{(0)}$ of $\delta X = \tilde{B} X$.

Discrete spirals of poles of $X_q^{(0)}$ condensate into cuts of $\tilde{X}^{(0)}$, continuous spirals $\tilde{z}_j q^{\mathbf{R}}$, where $\tilde{z}_0 = 1$ and $\tilde{z}_1, \ldots, \tilde{z}_m$ are the poles of $\tilde{B}$ on $\mathbf{C}^*$. The same holds at ∞.



Domain of definition of $\tilde{P}$

The connection matrix $P_q = \left(X_q^{(\infty)}\right)^{-1} X_q^{(0)}$ converges to $\tilde{P} = \left(\tilde{X}^{(\infty)}\right)^{-1} \tilde{X}^{(0)}$, which is locally constant on the open set: $\tilde{\Omega} = \mathbf{C}^* \setminus \bigcup \tilde{z}_j q^{\mathbf{R}}$, with values $\tilde{P}_1, \ldots, \tilde{P}_m$.

Theorem. - The monodromy matrix around $\tilde{z}_j$ *is* $\tilde{P}_j^{-1} \tilde{P}_{j-1}$.

II.3 Local Galois groupoid

[Sauloy 2003]

Let $\mathcal{E}^{(0)} = DiffMod(\mathbf{C}(\{z\}), \sigma_q)$ and $\mathcal{E}_f^{(0)}$ (resp. $\mathcal{P}^{(0)}$) the full subcategory of fuchsian systems (resp. with constant coefficients). Then, $\mathcal{P}^{(0)}$ is equivalent to $\mathcal{E}_f^{(0)}$.

We define a functor from $\mathcal{P}^{(0)}$ to the category $Fib_p(\mathbf{E}_q)$ of flat holomorphic vector bundles over $\mathbf{E}_q$ by putting:

$$\forall A \in GL_n(\mathbf{C}) \ , \ F_A = \frac{\mathbf{C}^* \times \mathbf{C}^n}{(z, X) \sim_A (qz, AX)} \to \mathbf{C}^*/q^{\mathbf{Z}} = \mathbf{E}_q,$$

Theorem. - We thus obtain a tannakian equivalence from $\mathcal{E}_f^{(0)}$ to $Fib_p(\mathbf{E}_q)$ (compare to [Baranovsky & Ginzburg]).

There is a family of fibre functors $(\omega_a^{(0)})_{a \in \mathbf{C}^*}$. We write $\mathbf{Z}^{alg} = Hom_{grp}(\mathbf{C}^*, \mathbf{C}^*) \times \mathbf{C}$ the proalgebraic hull of $\mathbf{Z}$.

Theorem. - The local Galois groupoid, with base $\mathbf{C}^*$, is given by: $Iso^{\otimes}(\omega_a^{(0)}, \omega_b^{(0)}) = \{(\gamma, \lambda) \in \mathbf{Z}^{alg} \ / \ \gamma(q)a = b\}$.

Corollary. - $G_f^{(0)} = \{(\gamma, \lambda) \in \mathbf{Z}^{alg} \ / \ \gamma(q) = 1\}$ is the local Galois group (compare to [Baranovsky & Ginzburg]).

II.4 Global Galois groupoid

[Sauloy 2003]

Let $\mathcal{E} = DiffMod(\mathbf{C}(z), \sigma_q)$ and $\mathcal{E}_f$ the full subcategory of fuchsian systems (at 0 and ∞). To A in $\mathcal{E}_f$, we associate (non canonically) $A^{(0)}, A^{(\infty)} \in GL_n(\mathbf{C})$ and $F \in GL_n(\mathcal{M}(\mathbf{C}^*))$ such that $(\sigma_q F)A^{(0)} = A^{(\infty)}F$.

More geometrically: two flat holomorphic vector bundles $F^{(0)}$ and $F^{(\infty)}$ over $\mathbf{E}_q$ and a *meromorphic* isomorphism $\phi : F^{(0)} \to F^{(\infty)}$; this is an intrinsic version of Birkhoff matrix, but **its formation is tensor compatible**.

Theorem. - (i) We thus obtain a tannakian equivalence from $\mathcal{E}_f$ to the category of such triples $(F^{(0)}, \phi, F^{(\infty)})$.
(ii) $\forall a \in \mathbf{C}^*$, $\left[(F^{(0)}, \phi, F^{(\infty)}) \rightsquigarrow F(a)\right] \in Iso^{\otimes}(\omega_a^{(0)}, \omega_a^{(\infty)})$.
(iii) Together with $G_f^{(0)}$ and $G_f^{(\infty)}$, these elements Zariski-generate the global Galois groupoid.

A is regular if $A(0) = A(\infty) = I_n$ (up to equivalence).

Corollary. - The Galois group of a regular system with connection matrix P is generated by the values $P(a)^{-1}P(b)$ (compare to [Etingof]).

II.5 Monodromy

[Sauloy 2003]

1. Local monodromy. - The semi-simple component $G_f^{(0)}$ is the Zariski closure of the free abelian group generated by: $\gamma_1 : ue^{-2\imath\pi\tau y} \mapsto u$ and $\gamma_2 : ue^{-2\imath\pi\tau y} \mapsto e^{2\imath\pi y}$, where $q = e^{-2\imath\pi\tau}$, $|u| = 1$ and $y \in \mathbf{R}$.

   We see γ_1, γ_2 as generating the (semi-simple component of the) π_1 of an infinitesimal elliptic curve, the local monodromy.

2. Global monodromy for abelian regular equations. - For an abelian systems, we get a rational map from an elliptic curve to a commutative linear group: geometric class field theory then provides an explicit description.

3. Confluency of the monodromy group. - The limit behavior of generators and relations can be described explicitly in relation with a differential Galois group (compare to [André]).

III.1 The canonical filtration by the slopes

[Sauloy 2004]

Here, $K = \mathbf{C}(\{z\})$. Formalization means: $- \otimes \mathbf{C}((z))$

Adams' Lemma. - Solutions built from the first slope are convergent.

Theorem. - Any q-difference module admits a unique filtration $(M^{\geq \mu})_{\mu \in \mathbf{Q}}$ with $M^{(\mu)} = \frac{M^{\geq \mu}}{M^{>\mu}}$ pure of slope μ. The filtration is functorial and $gr : M \rightsquigarrow \bigoplus M^{(\mu)}$ is a faithful exact $\mathbf{C}$-linear $\otimes$-compatible functor. After formalization, gr becomes isomorphic to the identity functor.

From now on, we consider only modules with integral slopes. Let $\mathcal{E}_1^{(0)}$ (resp. $\mathcal{E}_{mi,1}^{(0)}$) be the full subcategory of $\mathcal{E}^{(0)}$ made of modules (resp. direct sums of pure modules) with integral slopes. They are tannakian.

Theorem. - The Galois group $G_1^{(0)}$ of $\mathcal{E}_1^{(0)}$ is the semi-direct product of the Galois group $G_{mi,1}^{(0)} = \mathbf{C}^* \times G_f^{(0)}$ of $\mathcal{E}_{mi,1}^{(0)}$ by a (pro-)unipotent group St.

Problem : to build explicit galoisian Stokes operators generating the Stokes group St.

III.2 Isoformal and isograded classes

A formal class is given by a sum $M_0 = P_1 \oplus \cdots \oplus P_k$ of pure modules with slopes $\mu_1 > \cdots > \mu_k$ and ranks $r_1, \ldots, r_k$.

An analytic class within this formal class consists in a module M plus an isomorphism $gr(M) \to M_0$, up to an obvious equivalence relation.

Concretely, M has matrix:

$$A_U = \begin{pmatrix} z^{-\mu_1} A_1 & \ldots & \ldots & \ldots \\ \ldots & \ldots & U_{i,j} & \ldots \\ \ldots & 0 & \ldots & \ldots \\ 0 & \ldots & \ldots & z^{-\mu_k} A_k \end{pmatrix}, \text{ where } A_i \in GL_{r_i}(\mathbf{C}).$$

M_0 has the corresponding matrix A_0 (with $U = 0$).

Call $\mathfrak{G}$ the subgroup of GL_n made up of matrices:

$$\begin{pmatrix} I_{r_1} & \ldots & \ldots & \ldots \\ \ldots & \ldots & F_{i,j} & \ldots \\ \ldots & 0 & \ldots & \ldots \\ 0 & \ldots & \ldots & I_{r_k} \end{pmatrix}$$

There is a unique $\hat{F} \in \mathfrak{G}(\mathbf{C}((z)))$ such that $\hat{F}[A_0] = A_U$; call it $\hat{F}(U)$.

Lemma. - Sending A_U to $\hat{F}(U)$ induces a one-to-one correspondence between the set $\mathcal{F}(M_0)$ and a subset of the left quotient $\mathfrak{G}(\mathbf{C}(\{z\})) \backslash \mathfrak{G}(\mathbf{C}((z)))$.

III.3 Local analytical classification

[Ramis, Sauloy & Zhang]

Theorem (Birkhoff-Guenther normal form). - The set $\mathcal{F}(M_0)$ *of isoformal analytic classes can be parameterized by polynomial matrices with prescribed degrees:*

$$\forall i, j \text{ s.t. } 1 \leq i < j \leq k \text{ , coeffs}(U_{i,j}) \in \sum_{\mu_j \leq d < \mu_i} \mathbf{C} z^d.$$

(compare to [Birkhoff & Guenther]). Therefore, it is an affine algebraic variety of dimension $\sum_{i<j} r_i r_j (\mu_i - \mu_j)$.

The semigroup $\Sigma = q^{-\mathbf{N}}$ acts on $\mathbf{C}^*$ with horizon the elliptic curve $\mathbf{E}_q = \mathbf{C}^*/\Sigma$. There is an adapted theory of asymptotic developments.

Call $\Lambda_I(M_0)$ the sheaf of automorphisms of M_0 infinitely tangent to identity.

Theorem (q-analog of Malgrange-Sibuya). - There is a natural correspondance: $\mathcal{F}(M_0) \simeq H^1(\mathbf{E}, \Lambda_I(M_0))$.

There are two proofs:

1. Sheaf theoretic, using Newlander-Nirenberg theorem; this yields a more general result of Malgrange-Sibuya type.

2. Analytic, using discrete summation process for q-Gevrey divergent series; this yields an explicit construction of cocycles.

III.4 Algebraic summation

[Sauloy 2005]

An allowed summation divisor for A_0, is a family $(D_{i,j})_{1\leq i<j\leq k}$ of effective divisors over $\mathbf{E}_q$, such that deg $D_{i,j} = \mu_i - \mu_j$, $D_{i,j} = D_{i,l} + D_{l,j}$ and the evaluation $ev_{\mathbf{E}_q}(D_{i,j})$ does not belong to the image of $\overline{(-1)^{\mu_{i,j}}\frac{Sp(A_i)}{Sp(A_j)}}$ in $\mathbf{E}_q$.

Theorem. - For each A_U, there is a unique $F \in \mathfrak{G}(\mathcal{M}(\mathbf{C}^*))$ such that $F[A_0] = A_U$ and div$(F_{i,j}) \geq -D_{i,j}$. It is asymptotic to $\hat{F}(U)$.

Corollary. - The sheaf of solutions of A_U is locally isomorphic to the sheaf of solutions of of A_0. Hence, it is a vector bundle over $\mathbf{E}_q$.

For such an allowed summation divisor D, we write $F_D(U)$ for the F of the theorem and we see it as the summation of $\hat{F}(U)$ in direction D. It can either be obtained by algebraic algorithms, or by a transcendental discrete summation process as in [Ramis, Sauloy & Zhang].

Theorem. - The Stokes sheaf $\Lambda_I(M_0)$ has a devissage by pure *vector bundles.*

Here, pure bundle = line bundle $\otimes$ flat bundle.

Corollary. - A new proof of $\mathcal{F}(M_0) \simeq H^1(\mathbf{E}, \Lambda_I(M_0))$.

III.5 Galoisian Stokes operators

[Sauloy xxxx]

One can extend the fibre functor $\omega^{(0)}$ on $\mathcal{E}_f^{(0)}$ to $\mathcal{E}_{mi,1}^{(0)}$ by sending a pure module of integral slope to a pure vector bundle. We get a fibre functor $\widehat{\omega}^{(0)} = \omega^{(0)} \circ gr$ on $\mathcal{E}_1^{(0)}$.

We can also extend the fibre functor $\omega^{(0)}$ to $\mathcal{E}_1^{(0)}$ by sending a module M to the vector bundle of its solutions.

If one restricts to the essential subcategory $\mathcal{P}_1^{(0)}$ of objects in Birkhoff-Guenther normal form, the latter can be obtained by the same construction as on $\mathcal{P}^{(0)}$:

$$\frac{\mathbf{C}^* \times \mathbf{C}^n}{(z,X) \sim_A (qz, A(z)X)} \to \mathbf{C}^*/q^{\mathbf{Z}} = \mathbf{E}_q.$$

For each allowed summation divisor D, $A_U \rightsquigarrow F_D(U)$ is a natural transformation ϕ_D from $\widehat{\omega}^{(0)}$ to $\omega^{(0)}$.

Lemma. - For almost all $a \in \mathbf{E}_q$, putting $D_{i,j} = (\mu_i - \mu_j)a$ defines an allowed summation divisor; the corresponding natural transformation ϕ_a is $\otimes$-compatible.

Thus $\phi_{a,a'} = \phi_{a'} \circ \phi_a^{-1}$ is a $\otimes$-automorphism of $\omega^{(0)}$: these are galoisian Stokes operators.

Theorem. - The $\phi_{a,a'}$ generate the Stokes group St.

The vector bundle $\omega^{(0)}(M)$ admits a geometric construction without any condition on the slopes, as:

$$\frac{\overset{\circ}{D} \times \mathbf{C}^n}{(z, X) \sim_A (qz, A(z)X)} \to \mathbf{C}^*/q^{\mathbf{Z}} = \mathbf{E}_q,$$

where $\overset{\circ}{D}$ is any punctured disk of center 0 that does not meet the singular locus of $A(z)$ (and the relation $\sim_A$ is partial).

One can even replace $\overset{\circ}{D}$ by any open subset $U \in \mathbf{C}^*$ which projects onto $\mathbf{E}_q$ and such that $U \cap q^{-1}U$ does not meet the singular locus of $A(z)$. This features a *geometry of big annuli* and allows one to "localize" the singularities of A.

On the side of classification:

1. The functor $\omega^{(0)}$ is essentially surjective but not fully faithful. The slopes of a module have no meaning on the side of the bundle (unrelated to Harder-Narasimhan classification).

2. The functor (q-difference module) $\rightsquigarrow$ (vector bundle + filtration with pure quotient bundles) is fully faithful: it remembers the q-Gevrey filtration; but the image is unclear (probably related to Atiyah classification).

REFERENCES

André Y., 2001. Différentielles non-commutatives et théorie de Galois différentielle ou aux différences, *Ann. Scient. Ec. Norm. Sup (4)*, 34, no 5, 685–739.

Baranovsky V. and Ginzburg V., 1996. Conjugacy Classes in Loop Groups and G-Bundles on Elliptic Curves, International Mathematics Research Notes, no 15.

Birkhoff G.D., 1913. The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations, *Proc. Amer. Acad.*, 49, pp. 521-568.

Birkhoff G.D. and Guenther P.E., 1941. Note on a Canonical Form for the Linear q-Difference System, *Proc. Nat. Acad. Sci.*, Vol. 27, No. 4, pp. 218-222.

Etingof P.I., 1995. Galois Groups and Connection Matrices of q-difference Equations, *Electronic Research Announcements of the A.M.S.*, Vol. 1, Issue 1.

van der Put M. and Singer M.F., 1997. *Galois theory of difference equations*, *Lecture Notes in Mathematics*, 1666, Springer.

Ramis J.P., 1990. Fonctions θ et équations aux q-différences, unpublished, Strasbourg.

Ramis J.-P., Sauloy J. and Zhang C. Local analytic classification of irregular q-difference equations, *Article in preparation*.

Sauloy J., 2000. Systèmes aux q-différences singuliers réguliers : classification, matrice de connexion et monodromie, *Ann. Inst. Fourier*, 50-4 , 1021-1071.

Sauloy J., 2003. Galois theory of fuchsian q-difference equations, *Ann. Sci. École Norm. Sup. (4)*, 36, no 6, 925-968.

Sauloy J., 2004. La filtration canonique par les pentes d'un module aux q-différences et le gradué associé, *Ann. Inst. Fourier*, 54-1, 185-215.

Sauloy, 2005. Algebraic construction of the Stokes sheaf, to appear in *in Proceedings of the Ramis Conference, Asterisque*

Sauloy, xxxx. Irregular Galois theory, *work in progress*.