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speaking on

Topological models for p-local structures of finite groups

Algebraic approach to p-local structure of a finite group:

Definition (Puig). *Let G be a finite group and S be a Sylow subgroup of G. The fusion system of G is the category $\mathcal{F}_S(G)$, whose objects are the subgroups of S,*

$$ob\mathcal{F} = \{P \leq S\},$$

and whose morphism sets are given by

$$Hom_{\mathcal{F}(G)}(P, Q) = Hom_G(P, Q)$$

for all p-subgroups $P, Q \leq S$.

Topological approach: $BG_p^{\wedge}$

By the Martino-Priddy conjecture, these two approaches are equivalent.

Theorem (Martino-Priddy,Oliver). *Let G and G' be finite groups with p-Sylow subgroups S and S', respectively. Then*

$$\mathcal{F}_S(G) \cong \mathcal{F}_{S'}(G') \Leftrightarrow BG^\wedge_p \simeq BG'^\wedge_p.$$

Proof. "$\Leftarrow$:" Proved by Martino and Priddy ('96) using a homotopy theoretic construction.

"$\Rightarrow$:" Proved by Oliver ('01/'02) by showing vanishing of obstructions to uniqueness of classifying spaces. Uses classification of finite simple groups. $\square$

This allows us to regard $BG^\wedge_p$ as a classifying space for $\mathcal{F}_S(G)$.

Puig formalised fusion systems as follows:

Definition (Puig). *Let S be a finite p-group. A fusion system over S is a category $\mathcal{F}$, whose objects are the subgroups of S,*

$$ob\mathcal{F} = \{P \leq S\},$$

and whose morphism sets satisfy the following conditions

(a) $Hom_S(P, Q) \subseteq$ *Hom*$_{\mathcal{F}}(P, Q) \subseteq Inj(P, Q)$ *for all $P, Q \leq S$.*

(b) *Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.*

Puig identified two important properties enjoyed by fusion systems of groups. Fusion systems with these properties are called *saturated fusion systems*.

Condition (I) is a “prime-to-p condition”.
Condition (II) is a “maximal extension” condition.

Saturated fusion systems arise naturally in contexts besides finite groups.
Examples:

- In p-modular representation theory, induced by Brauer subpairs of blocks of group algebras.

- As Chevalley groups of p-compact groups.

It would be useful to have a functorial assignment of classifying spaces to saturated fusion systems.

(Interlude:)
Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be fusion systems over finite p-groups S_1 and S_2, respectively.

A morphism between fusion systems

$$(S_1, \mathcal{F}_1) \to (S_2, \mathcal{F}_2)$$

is not just a functor

$$F \colon \mathcal{F}_1 \to \mathcal{F}_2.$$

We demand that it be compatible with a group homomorphism

$$\gamma \colon S_1 \to S_2,$$

so that

$$F(P) = \gamma(P),$$

for all $P \leq S_1$ and

$$\gamma|_Q \circ \varphi = F_\gamma(\varphi) \circ \gamma|_P$$

for all $\varphi \in \mathrm{Hom}_{\mathcal{F}_1}(P, Q)$.

In this case γ actually determines S. We say γ is a $(\mathcal{F}_1, \mathcal{F}_2)$-*fusion-preserving homomorphism*. Morphisms between fusion systems are the fusion preserving homomorphisms.

Broto-Levi-Oliver have developed a framework for classifying spaces of saturated fusion systems. They define a *p-local finite group* as a triple $(S, \mathcal{F}, \mathcal{L})$, where:

- S is a finite p-group,
- $\mathcal{F}$ is a saturated fusion system over S,
- $\mathcal{L}$ is an associated centric linking system.

A centric linking system associated to $\mathcal{F}$ is a category $\mathcal{L}$ sitting over a full subcategory $\mathcal{F}^c$ of $\mathcal{F}$. It comes equipped with a functor

$$\pi\colon \mathcal{L} \to \mathcal{F}^c$$

which is the identity on objects and surjective on morphisms.

The *classifying space* of $(S, \mathcal{F}, \mathcal{L})$ is $|\mathcal{L}|_p^\wedge$.

The use of this term is justified by the following two facts:

1. $(S, \mathcal{F}, \mathcal{L})$ can be reconstructed from $|\mathcal{L}|_p^\wedge$ by a homotopy-theoretic construction.

2. When $\mathcal{F} = \mathcal{F}_S(G)$, Oliver's proof of the Martino-Priddy conjecture shows that $|\mathcal{L}|_p^\wedge \simeq BG_p^\wedge$.

Broto-Levi-Oliver have proved many satisfying results about p-local finite groups, extending known results for finite groups. Some examples follow.

Cohomology: The cohomology of the classifying space is

$$H^*(|\mathcal{L}|_p^\wedge; \mathbf{F}_p) = H^*(\mathcal{F}) := \varprojlim_{\mathcal{F}} H^*(B(-); \mathbf{F}_p),$$

the ring of invariants under $\mathcal{F}$-conjugacy in $H^*(BS; \mathbf{F}_p)$.

Mapping spaces: If Q is any p-group, then

$$[BQ, |\mathcal{L}|_p^\wedge] \cong \mathrm{Hom}\,(Q, S)\,/(\mathcal{F} - \text{conjugacy}).$$

They also give homotopy decompositions of classifying spaces and describe spaces of self equivalences.

In general it is not known whether a given saturated fusion system has an associated centric linking system, and if so whether it is unique.

Broto-Levi-Oliver have developed an obstruction theory to address this question and settled some special cases.

Oliver: The fusion system of a group has a unique associated centric linking system.

Broto-Levi-Oliver: A saturated fusion system over a p-group of rank ≤ 3 has an associated centric linking system. If the rank is ≤ 2, then that centric linking system is unique.

Even when restricting to saturated fusion systems with a unique centric linking system, the assignment of classifying spaces is not known to be functorial.

The problem is that a fusion preserving homomorphism

$$(S_1, \mathcal{F}_1) \to (S_2, \mathcal{F}_2)$$

need not map the subcategory $\mathcal{F}_1^c$ to $\mathcal{F}_2^c$.

These problems can be addressed stably.

Based on ideas by Linckelmann-Webb, Broto-Levi-Oliver construct a *classifying spectrum* $\mathbb{B}\mathcal{F}$ for a saturated fusion system $\mathcal{F}$ over a finite p-group S.

This is, up to homotopy, the unique summand of $\Sigma^\infty BS$ such that

$$H^*(\mathbb{B}\mathcal{F}; \mathbf{F}_p) \cong H^*(\mathcal{F}).$$

If $\mathcal{L}$ is an associated centric linking system one has

$$\mathbb{B}\mathcal{F} \simeq \Sigma^\infty |\mathcal{L}|^\wedge_p.$$

In my work, I have refined the B-L-O construction, and endow the classifying spectrum with a *structure map*

$$\sigma_{\mathcal{F}}\colon \Sigma^{\infty}BS \to \mathbb{B}\mathcal{F}.$$

This allows me to prove the following results:

Theorem (KR). *The assignment*

$$\mathcal{F} \longrightarrow \mathbb{B}\mathcal{F}$$

is functorial.

That is, a fusion preserving homomorphism induces a map of classifying spectra.

Theorem (KR). *A saturated fusion system $\mathcal{F}$ can be recovered from the structure map*

$$\sigma_{\mathcal{F}}\colon \Sigma^\infty BS \to \mathbb{B}\mathcal{F}$$

by a homotopy-theoretic construction.

When applied to fusion systems of groups, this gives an alternative to Martino-Priddy's "Stable classification of $BG_p^\wedge$".

This result is surprising since Martino-Priddy have shown that the homotopy type of $\mathbb{B}\mathcal{F}$ alone does not determine $\mathcal{F}$. It *must* be regarded as an object under $\Sigma^\infty BS$.

Theorem (KR). *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be saturated fusion systems over finite p-groups S_1 and S_2, respectively. Then*

$$[\mathbb{B}\mathcal{F}_1, \mathbb{B}\mathcal{F}_2] \cong (\mathcal{F}_1\text{–}\mathit{conj})\backslash\{BS_1, BS_2\}/(\mathcal{F}_2\text{–}\mathit{conj}).$$

When applied to fusion systems of groups, this gives a new variant of the Segal conjecture, describing stable maps between p-completed classifying spaces of finite groups in the same simple terms as stable maps between classifying spaces of finite p-groups.