

Cayley graphs and totally disconnected locally compact groups

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Compactly generated totally disconnected locally compact groups

- Totally disconnected – the only connected subsets are singletons.
- Locally compact – every point has an open neighbourhood with compact closure.
- Compactly generated – compact generating set in the algebraic sense.

Where do we get totally disconnected groups from:

- Let G be a topological group and G_0 the connected component of the identity element. The group G/G_0 is totally disconnected.
- Every group is a totally disconnected locally compact group (use the discrete topology).
- Lie-groups over the p -adic numbers.
- Automorphism groups of locally finite connected graphs.
- Pro-finite groups.

General aim

Geometric theory of totally disconnected locally compact groups.

Should relate to/include parts of ordinary geometric group theory and also parts from the theory of Lie-groups.

The topic of this talk is related to a project based on the work of G. A. Willis, Newcastle Australia.

Groups and graphs

An action of a group on a graph used to treat and clarify the structure of a group.

- Cayley graphs
- Bass-Serre theory of groups acting on trees
- Finite permutation groups
- Finite simple groups

v. Dantzig's Theorem

For a long time the “only” general result known about totally disconnected locally compact groups was:

Theorem 1 (*v. Dantzig, 1936*) *A totally disconnected locally compact group always contains a compact open subgroup. Indeed, there is a neighbourhood basis of the identity consisting of compact open subgroups.*

Now we also have the structure theory of Willis.

Good generating sets

Lemma 1 *Let G be a compactly generated totally disconnected locally compact group. Let U be a compact open subgroup of G . Then there is a finite set $T = \{h_1, \dots, h_n\}$ closed under taking inverses such that $H = \langle h_1, \dots, h_n \rangle$ acts transitively on the set of right cosets G/U . Furthermore, every element in G can be written as $h_{i_1} h_{i_2} \cdots h_{i_n} u$ where $u \in U$*

Definition 1 Let G be a compactly generated totally disconnected locally compact group. A compact open subgroup U together with a finite set T as described in the Lemma above is said to form a *good* generating set.

Rough Cayley graphs

Let G be a compactly generated totally disconnected locally compact group.

Suppose U is a compact open subgroup and $T = \{h_1, \dots, h_n\}$ a finite set such that $T^{-1} = T$ and U and T together form a good generating set.

The rough Cayley graph $\Gamma = \text{RCay}(G, U, T)$ is defined such that

- the vertex set is $V\Gamma = G/U$
- two elements gU and $g'U$ are adjacent in Γ if

$$gU = h_{i_1} \cdots h_{i_m} U \text{ and } g'U = h_{i_1} \cdots h_{i_m} h_{i_{m+1}} U$$

Theorem 2 *The graph $\Gamma = \text{RCay}(G, U, T)$ is connected, locally finite, and G acts transitively on Γ .*

Rough Cayley graphs

– alternative definition –

Let G be a compactly generated totally disconnected locally compact group.

Suppose U is a compact open subgroup and $T = \{h_1, \dots, h_n\}$ a finite set such that $T^{-1} = T$ and U and T together form a good generating set.

The rough Cayley graph $\Gamma = \text{RCay}(G, U, T)$ is defined such that

- the vertex set is $V\Gamma = G/U$
- Let α be some vertex in Γ (convenient to think of α as the coset U). The edge set of Γ is

$$E\Gamma = G\{\alpha, h_1\alpha\} \cup \dots \cup G\{\alpha, h_m\alpha\}.$$

Rough Cayley graphs

– quasi-isometries –

The choice of a compact open subgroup and the finite set T does not matter all that much.

Theorem 3 *Let G be a compactly generated totally disconnected locally compact group. Suppose U and V are compact open subgroups of G such that U and V together with T and S , respectively, form good generating sets. Then the rough Cayley graphs $\text{RCay}(G, U, T)$ and $\text{RCay}(G, V, S)$ are quasi-isometric to each other.*

Properties of $\text{RCay}(G, U, T)$ that are invariant under quasi-isometries can be regarded as properties of the group G .

Ends

The ends of the graph $\text{RCay}(G, U, T)$ can be regarded as an invariant of G , called the *rough ends* of G . Can also be used to compactify G .

We get an analogue of Stallings End's theorem

Theorem 4 (Cf. Abels, 1974) *Let G be a compactly generated totally disconnected locally compact group. Suppose that some (equivalently, any) rough Cayley graph G has infinitely many ends. Then the group G splits over some compact open subgroup, more precisely, $G = A *_C B$ or $G = A *_C x$ where A and B are closed open compactly generated subgroups of G and C is a compact open subgroup.*

Two ended compactly generated totally disconnected locally compact groups are characterized by:

Theorem 5 (Cf. Möller and Seifert, 1998) *Let G be a compactly generated totally disconnected locally compact group. Suppose that the space of rough ends has precisely two points. Then G has a compact open normal subgroup N such that G/N is either isomorphic to \mathbb{Z} or D_∞ .*

Accessibility

Definition 2 A finitely generated group is said to be *accessible* if it has an action on a tree T such that:

- (i) the number of orbits of G on the edges of T is finite;
- (ii) the stabilizers of edges in T are finite subgroups of G ;
- (iii) every stabilizer of a vertex in T is a finitely generated subgroup of G and has at most one end.

Definition 3 (Thomassen and Woess, 1993) Let Γ be a connected locally finite graph. If there is a number k such that any two distinct ends can be separated by removing k or fewer vertices from Γ then the graph Γ is said to be *accessible*.

Theorem 6 (Thomassen and Woess, 1993) *A finitely generated group is accessible if and only if every Cayley graph with respect to a finite generating set is accessible.*

Accessibility

– compactly generated totally disconnected –

– locally compact group –

Definition 4 A compactly generated totally disconnected locally compact group is said to be *accessible* if it has an action on a tree T such that:

(i) the number of orbits of G on the edges of T is finite;

(ii) the stabilizers of edges in T are compact open subgroups of G ;

(iii) every stabilizer of a vertex in T is a compactly generated subgroup of G and its rough Cayley graphs have at most one end.

Theorem 7 *Let G be a compactly generated totally disconnected locally compact group. Then G is accessible if and only if every rough Cayley graph of G is accessible.*

Polynomial growth

Definition 5 Let G be a locally compact group generated by a compact symmetric neighbourhood of the identity V . Set $V^n = \{g_1 g_2 \cdots g_n \mid g_i \in V\}$. Let μ denote a Haar measure on G . If there are constants c and d such that $\mu(V^n) \leq cn^d$ for all natural numbers n then we say that G has polynomial growth.

Theorem 8 *Let G be a compactly generated totally disconnected locally compact group and Γ some rough Cayley graph for G . Then Γ has polynomial growth if and only if G has polynomial growth (in the sense of Definition 5).*

Theorem 9 (Cf. Gromov, 1981, and Trofimov, 1985) *Let G be a compactly generated totally disconnected locally compact group. Then G has polynomial growth if and only if G has a normal compact open subgroup K such that G/K is a finitely generated almost nilpotent group.*

Other uses

– FC^- -elements –

Definition 6 An element g in a topological group is said to be an FC^- -*element* if the conjugacy class of g has compact closure in G .

The FC^- -elements in G form a normal subgroup of G . Has been investigated by numerous authors.

By using a result of Trofimovs about automorphism groups of graphs one can prove

Theorem 10 (Möller, 2003) *Let G be a totally disconnected locally compact group. Assume furthermore that G is compactly generated. Then the subgroup of FC^- -elements in G is closed in G .*

The subgroup of FC^- -elements in G need not be closed if G is not compactly generated.

Other uses

– Willis's structure theory –

Graph theoretical constructions like described above can be used to clarify and treat concepts from Willis's structure theory of totally disconnected locally compact groups.

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