Probabilistic Symmetries and Invariance Principles*

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*Also the title of a new book, soon to be published by Springer-Verlag, NY

Notes added at submission:

- Many transparencies shown during my talk were simple sketches of processes etc. Those are not included here.

- Much of what I said during the talk is not included on the transparencies. In many cases, this additional discussion is needed to make sense of the present text.

- Some statements shown here are slightly simplified to fit the format of the transparencies. For the technically precise versions, you need to consult my forthcoming book.

- As emphasized in my talk, the purpose of my presentation was only to give an introduction to some of the basic, underlying ideas, set in their historical context. Most results exhibited here represent only the starting point of the systematic theory, as developed in my book.

Basic Symmetries

invariant objects	*transformations*
stationary	shifts
contractable	subsequences
exchangeable	permutations
rotatable	isometries

Symmetries on Sequences

♣ *Stationarity:*

$$(\xi_1, \xi_2, \ldots) \stackrel{d}{=} (\xi_2, \xi_3, \ldots)$$

♣ *Contractability:*

$$(\xi_1, \xi_2, \ldots) \stackrel{d}{=} (\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots)$$

♣ *Exchangeability:*

$$(\xi_1, \xi_2, \ldots) \stackrel{d}{=} (\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \xi_k, \xi_{k+2}, \ldots)$$

♣ *Rotatability:*

$$(\xi_1, \xi_2, \ldots) \stackrel{d}{=} U(\xi_1, \xi_2, \ldots)$$

We say that $\xi_1, \xi_2, \ldots$ are

— *i.i.d.* if they are independent with a common distribution μ

— *mixed i.i.d.* if their joint distribution is a mixture of i.i.d. distributions: First pick a measure μ at random, then choose the ξ_k to be independent with distribution μ

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♣ **de Finetti (1931, 1937)** *An infinite sequence of random variables* $\xi_1, \xi_2, \ldots$ *is exchangeable iff it is mixed i.i.d.*

♣ **Ryll-Nardzewski (1957)** *An infinite sequence of random variables is contractable iff it is exchangeable, hence mixed i.i.d.*

Continuous-Time Transformations

♣ *Reflections:*

$$R_a(t) = \begin{cases} a - t, & t \leq a, \\ t, & t > a. \end{cases}$$

♣ *Contractions:*

$$C_{a,b}(t) = \begin{cases} t, & t \leq a, \\ \infty, & t \in (a,b), \\ t - b + a, & t > b. \end{cases}$$

♣ *Transpositions:*

$$T_{a,b}(t) = \begin{cases} t + b - a, & t \leq a, \\ t - a, & t \in (a,b), \\ t, & t > b. \end{cases}$$

♣ *For an rcll process X on $\mathbb{R}_+$, it is equivalent that X be contractable, exchangeable, or reflectable.*

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Now say that a process X is

— *Lévy* (named after Paul Lévy) if it has stationary, independent increments and its paths are right-continuous with left-hand limits (*càdlàg* or *rcll*)

— *mixed Lévy* if its distribution is a mixture of Lévy-process distributions: First pick such a measure μ at random, then choose X to have distribution μ

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♣ **Bühlmann (1960)** *An rcll process X on $\mathbb{R}_+$ is exchangeable iff it is mixed Lévy.*

Finite Sequences, Intervals

An *urn sequence* is obtained by picking tickets at random from an urn. For a *mixed urn sequence*, first pick an urn at random, then the tickets.

♣ *A finite sequence of random variables* $\xi_1, \ldots, \xi_n$ *is exchangeable iff it is a mixed urn sequence.*

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♣ *An rcll process* X *on* $[0,1]$ *is exchangeable iff a.s.*

$$X_t = \alpha t + \sigma B_t + \textstyle\sum_k \beta_k (1\{\tau_k \le t\} - t),$$

for some independent sets of objects:

B	*Brownian bridge on* $[0,1]$
$\tau_1, \tau_2, \ldots$	*i.i.d., uniform r.v.'s*
α, σ, $\beta_1, \beta_2, \ldots$	*any random variables*

♣ *A continuous process X on $\mathbb{R}_+$ or $[0,1]$ is exchangeable iff a.s.*

$$X_t = \alpha t + \sigma B_t$$

for some Brownian motion or bridge B and an independent pair of random variables α and σ.

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♣ *A continuous process X on $\mathbb{R}_+$ is rotatable iff a.s.*

$$X_t = \sigma B_t$$

for some Brownian motion B and an independent random variable σ.

♣ A sequence $c_0, c_1, \ldots$ in $\mathbb{R}$ is said to be *completely monotone* if

$$(-1)^n \Delta^n c_k \geq 0, \quad k, n \geq 0.$$

♣ A function f on $\mathbb{R}_+$ or $[0,1]$ is said to be *completely monotone* if the sequence $f(nh)$ is completely monotone for every $h > 0$. It is essentially equivalent that f be C^∞ with

$$(-1)^n f^{(n)}(t) \geq 0, \quad t, n \geq 0.$$

♣ A complex-valued function f on $\mathbb{R}^d$ is said to be *non-negative definite* if

$$\sum_{i,j} c_i \bar{c}_j f(x_i - x_j) \geq 0,$$

for any $c_1, \ldots, c_d \in \mathbb{C}$ and $x_1, \ldots, x_d \in \mathbb{R}^d$.

♣ **Hausdorff (1921)** *A sequence* $c_0, c_1, \ldots$ *with* $c_0 = 1$ *is completely monotone iff*

$$c_k = E\alpha^k, \quad k \in \mathbb{Z}_+,$$

for some random variable α *in* $[0,1]$.

♣ **Bernstein (1928)** *A function* f *on* $\mathbb{R}_+$ *with* $f(0+) = 1$ *is completely monotone iff*

$$f(t) = Ee^{-\rho t}, \quad t > 0,$$

for some random variable $\rho \geq 0$.

♣ **Bernstein (1928)** *A function* f *on* $[0,1]$ *with* $f(0+) = 1$ *is completely monotone iff*

$$f(t) = E(1-t)^\kappa, \quad t \in (0,1],$$

for some random variable κ *in* $\mathbb{Z}_+$.

The Hausdorff/Bernstein theorems are equivalent to the following:

A point process on S is exchangeable iff it is

- ♣ *mixed Bernoulli when* $S = \mathbb{N}$
- ♣ *mixed Poisson when* $S = \mathbb{R}_+$
- ♣ *mixed binomial when* $S = [0,1]$

Here we define a

— *Bernoulli sequence* as a sequence of independent digits 0 and 1, with a fixed probability α for 1

— *Poisson process* as a point process with independent increments, constant rate $\rho \geq 0$

— *binomial process* as a process of κ independent, uniformly distributed points in $[0,1]$

For the corresponding mixed versions, first pick an α, ρ, or κ at random ...

♣ **Schoenberg (1938)** *A continuous function f on $\mathbb{R}_+$ with $f(0) = 1$ is completely monotone iff for every $n \in \mathbb{N}$ the function*

$$f_n(x_1, \ldots, x_n) = f(x_1^2 + \cdots + x_n^2)$$

is non-negative definite on $\mathbb{R}^n$.

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For a probabilistic counterpart, let $N(0,1)$ denote that standard normal (Gaussian) distribution on $\mathbb{R}$ with density

$$\varphi(x) = (2\pi)^{-1/2} e^{-x^2/2}$$

♣ **Freedman (1962)** *An infinite sequence of random variables $\xi_1, \xi_2, \ldots$ is rotatable iff a.s.*

$$\xi_k = \sigma\, \eta_k, \quad k \in \mathbb{N},$$

for some i.i.d. $N(0,1)$ random variables $\eta_1, \eta_2, \ldots$ and an independent random variable σ.

♣ **Maxwell (1875–78)** *Let* $\xi_1, \ldots, \xi_n$ *be independent, rotatable random variables. Then the* ξ_k *are i.i.d. centered Gaussian.*

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♣ **Poincaré/Borel (1912–14)** *For* $n \in \mathbb{N}$*, let* $\xi_{n1}, \ldots, \xi_{nn}$ *be rotatable random variables restricted to the sphere* $|x| = \sqrt{n}$ *in* $\mathbb{R}^n$*. Then for fixed* k *the variables* $\xi_{n1}, \ldots, \xi_{nk}$ *are asymptotically i.i.d.* $N(0,1)$ *as* $n \to \infty$*.*

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In a gas with a large number of molecules, the velocity vectors are independent, centered Gaussian.

A discrete random time τ is said to be *optional* (*non-anticipating* or a *stopping time*) if the event $\{\tau = k\}$ is determined by the history of all underlying processes up to time k. The *shift operators* θ_n are defined by

$$\theta_n(\xi_1, \xi_2, \ldots) = (\xi_{n+1}, \xi_{n+2}, \ldots).$$

Say that $\xi = (\xi_1, \xi_2, \ldots)$ is *stationary* if $\theta_n \xi \stackrel{d}{=} \xi$ for all $n \in \mathbb{N}$ and *strongly stationary* if $\theta_\tau \xi \stackrel{d}{=} \xi$ for every optional time $\tau < \infty$.

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For any infinite sequence of random variables $\xi = (\xi_1, \xi_2, \ldots)$*, these conditions are equivalent:*

- ♣ ξ *is exchangeable*
- ♣ ξ *is contractable*
- ♣ ξ *is strongly stationary*

A discrete random time τ is said to be *predictable* if the event $\{\tau = k\}$ is determined by the history of all underlying processes up to time $k - 1$ (so that the occurrence of τ can be predicted one step in advance).

Optional Skipping:

♣ **Doob (1936)** *Let* $\xi_1, \xi_2, \ldots$ *be i.i.d. random variables, and let* $\tau_1 < \tau_2 < \cdots$ *be predictable times in* $\mathbb{N}$. *Then*

$$(\xi_{\tau_1}, \xi_{\tau_2}, \ldots) \stackrel{d}{=} (\xi_1, \xi_2, \ldots). \qquad (1)$$

Predictable Sampling:

♣ *Let* $\xi_1, \xi_2, \ldots$ *be a finite or infinite, exchangeable sequence of random variables. Then* (1) *holds for any distinct, predictable times* $\tau_1, \tau_2, \ldots$

♣ In a casino you are watching a *roulette*, where only red and black may occur, each with probability $\frac{1}{2}$ (no slots for the bank). On the basis of what you have observed, at a suitable moment you decide to bet \$100 on red in the next round. If you win, you get the double amount back, otherwise you lose. Here your average gain is clearly zero, regardless of strategy, so the gambling is totally pointless.

♣ Now consider the corresponding game with a well-shuffled *card deck* (regular deck with 26 red and 26 black cards). You pick the cards one by one (without replacement). At a suitable time, based on your observations, you bet \$100 on the next card, and you win the double amount if the card is red. *How should you play to maximize your average gain?* Note that at every moment you know the proportion of red cards in the remaining deck.

An array $X = (X_{ij})$ on $\mathbb{N}^2$ is said to be

— *separately exchangeable* if $(X_{p_i,q_j}) \stackrel{d}{=} X$ for any permutations $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$ of $\mathbb{N}$

— *jointly exchangeable* if $(X_{p_i,p_j}) \stackrel{d}{=} X$ for any permutation $p_1, p_2, \ldots$ of $\mathbb{N}$

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Aldous/Hoover (1981-82) *An array X is*

♣ *separately exchangeable iff a.s.*

$$X_{ij} = f(\alpha, \xi_i, \eta_j, \zeta_{ij})$$

♣ *jointly exchangeable iff a.s.*

$$X_{ij} = f(\alpha, \xi_i, \xi_j, \zeta_{\{i,j\}})$$

for a measurable function f on $[0,1]^4$ and some i.i.d. $U(0,1)$ random variables α, ξ_i, η_j, ζ_{ij} or $\zeta_{\{i,j\}}$.

An array $X = (X_{ij})$ on $\mathbb{N}^2$ is said to be

— *separately rotatable* if $(U_1 \otimes U_2)X \overset{d}{=} X$ for any orthogonal matrices U_1 and U_2

— *jointly rotatable* if $U^{\otimes 2}X \overset{d}{=} X$ for any orthogonal matrix U

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♣ **Aldous (1981)** *An array $X = (X_{ij})$ is separately rotatable iff a.s.*

$$X_{ij} = \sigma\,\zeta_{ij} + \textstyle\sum_k \alpha_k\,\xi_{ik}\,\eta_{jk}$$

for some i.i.d. $N(0,1)$ r.v.'s $\xi_{ik}, \eta_{jk}, \zeta_{ij}$ and an independent set of coefficients σ, α_k.

♣ *An array X is jointly rotatable iff a.s.*

$$\begin{aligned} X_{ij} &= \rho\,\delta_{ij} + \sigma\,\zeta_{ij} + \sigma'\,\zeta_{ji} \\ &\quad + \textstyle\sum_{h,k} \alpha_{hk}\,(\xi_{ik}\,\xi_{jk} - \delta_{ij}\,\delta_{hk}) \end{aligned}$$

for some i.i.d. $N(0,1)$ r.v.'s ξ_{ik}, ζ_{ij} and an independent set of coefficients $\rho, \sigma, \sigma', \alpha_{hk}$.

Exchangeability and Contractability in Higher Dimensions

♣ **Hoover (1979)** *An array* $X = (X_k)$ *on* $\mathbb{N}^d$ *is jointly exchangeable iff a.s.*

$$X_k = f(\hat{\xi}_k) \qquad (2)$$

for a measurable function f *and some i.i.d.* $U(0,1)$ *random variables* ξ_J, $J \subset \mathbb{N}$ *with* $|J| \le d$.

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♣ *An array* $X = (X_k)$ *on the tetrahedral set* $\Delta_d = \{k_1 < \cdots < k_n\}$ *in* $\mathbb{N}^d$ *is jointly contractable iff it can be represented as in* (2).

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♣ *An array* $X = (X_k)$ *on* Δ_d *is jointly contractable iff it can be extended to a jointly exchangeable array on* $\mathbb{N}^d$.

♣ By a *continuous linear random functional (CLRF)* on a Hilbert space H we mean a process X on H (operator $X : H \to L^0(P)$) such that

$$X(ah + bk) = aXh + bXk \text{ a.s.},$$
$$Xh \xrightarrow{P} 0 \text{ as } h \to 0 \text{ in } H.$$

♣ An *isonormal Gaussian process (G-process)* on H is a centered Gaussian process ξ on H such that

$$E(\xi h\, \xi k) = \langle h, k \rangle, \quad h, k \in H.$$

♣ For some independent or identical G-processes $\xi_1, \ldots, \xi_d$ on H, the associated *multiple Wiener-Itô integral* $\otimes_k \xi_k$ is defined as the a.s. unique CLRF on $H^{\otimes d}$ such that a.s.

$$\left(\bigotimes_k \xi_k\right)\left(\bigotimes_k h_k\right) = \prod_k \xi_k h_k$$

for any (orthogonal) elements $h_1, \ldots, h_d \in H$.

Rotatability in Higher Dimensions

A CLRF X on $H^{\otimes d}$ is said to be

— *separately rotatable* if $X \circ \bigotimes_k U_k \stackrel{d}{=} X$ for any unitary operators $U_1, \ldots, U_d$ on H

— *jointly rotatable* if $X \circ U^{\otimes d} \stackrel{d}{=} X$ for any unitary operator U on H

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A CLRF X on H is

♣ *separately rotatable iff a.s.*

$$Xf = \sum_{\pi \in \mathcal{P}_d} \left(\bigotimes_{J \in \pi} \eta_J \right) (\alpha_\pi \otimes f)$$

♣ *jointly rotatable iff a.s.*

$$Xf = \sum_{\pi \in \mathcal{O}_d} \left(\bigotimes_{k \in \pi} \eta_{|k|} \right) (\alpha_\pi \otimes f)$$

for some independent G-processes η_J or η_k and an independent set of coefficients α_π.