Deformation theory of reflexive modules on rational surface singularities

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Outline

Introduction

Notation and basic facts

Let $X = \operatorname{Spec} \mathcal{O}_X$ be an affine scheme over an algebraically closed field k (of char. 0), where $\mathcal{O}_X$ is a local ring which is henselian and algebraic. We will assume that X is a normal surface singularity.

Fact

There exists a minimal resolution $\pi : \widetilde{X} \to X$ of the singularity X.

Definition

We say that X is a *rational* surface singularity if $\mathrm{R}^1\pi_*\mathcal{O}_{\widetilde{X}} = 0$.

Facts about rational surface singularities

- Let $E = (\widetilde{X} \times_X \operatorname{Spec} k)_{\mathrm{red}} \subseteq \widetilde{X}$. Then $E = \cup_{i=1}^n E_i$ where $E_i \cong \mathbb{P}^1_k$.
- $\operatorname{Pic} \widetilde{X} \cong \mathbb{Z}^n$ is generated by divisors D_i , $i = 1, \ldots, n$, with D_i transversal to E_i .
- There is an intersection theory on $\widetilde{X}$; the intersection form $\langle E_i, E_j \rangle$ is negative definite.

Definition

The resolution graph is the weighted graph with vertices corresponding to E_i and where there is an edge between E_i and E_j if $E_i \cap E_j \neq \emptyset$. The weight of a vertex is the self intersection number E_i^2.

Definition

The cokernel (as abelian group) of the intersection matrix $\langle E_i, E_j \rangle$ is denoted by H.

Theorem

If X is a rational surface singularity, then

$$\mathrm{Cl}(X) \cong H$$

and this is a finite group. Here $\mathrm{Cl}(X)$ denote the divisor class group of X, i.e. the free abelian group generated by height one prime ideals divided by principal divisors.

Rational double points

Definition

Let X be a rational surface singularity. We say that X is a rational double point (RDP) if X has multiplicity $m = 2$.

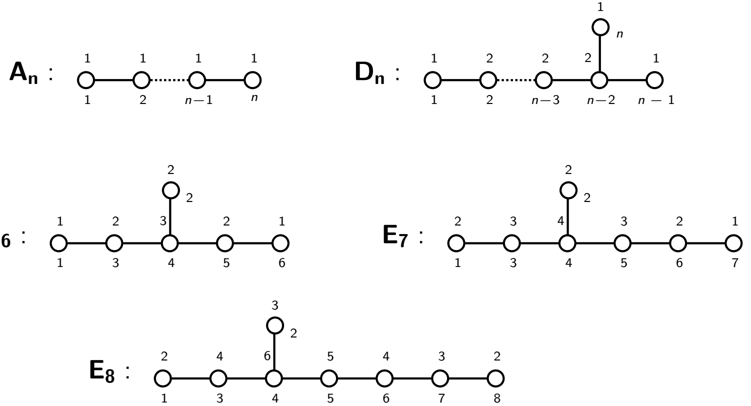
Theorem

Assume that X is a rational surface singularity. The following are equivalent.

- X is a rational double point
- X is Gorenstein
- X is a hypersurface
- X is a complete intersection
- the dual graph of X contains only vertices with weight -2
- X is a quotient of k^2 by a finite group $G \subset \mathrm{SL}(k, 2)$.

Rational double points

Resolution graphs



All weights are -2. The numbers below enumerate the vertices, and the numbers above give the multiplicity in the fundamental cycle.

Reflexive modules on rational surface singularities

Definition

A finitely generated $\mathcal{O}_X$-module M is called *reflexive* if the canonical map to its double $\mathcal{O}_X$-dual, $M \to M^{\vee\vee}$, is an isomorphism.

Facts

- When X is 2-dimensional and normal, a reflexive module is the same as a maximal Cohen-Macaulay module.
- M restricted to the regular locus $U \subseteq X$ is locally free.

Definition

Let $\widetilde{M} = \pi^* M/\text{torsion}$. ($\pi : \widetilde{X} \to X$ is the minimal resolution.) $\widetilde{M}$ is called a *full* sheaf.

Fact

The sheaf $\widetilde{M}$ is locally free and $M = \mathrm{H}^0(X, \pi_* \widetilde{M})$.

Reflexive modules on rational surface singularities

Facts

Assume X is a normal surface singularity.

- X is rational precisely if X has finitely many rank one indecomposable reflexive modules.
- X is a quotient singularity if X have finitely many indecomposable reflexive modules.

The McKay correspondence

Theorem

Assume that X is rational double point.

- Let M be a non-trivial indecomposable reflexive module on X. The Chern class $c_1(\widetilde{M}) \in \operatorname{Pic} \widetilde{X}$ is represented by a transversal divisor D_i to E_i, and this sets up a one to one correspondence between isomorphism classes of non-trivial, indecomposable reflexive modules on X and irreducible components of the exceptional set.
- $\operatorname{rank} M =$ the multiplicity of E_i in the fundamental cycle Z.

The versal deformation of a module

Definition

Let M be a coherent $\mathcal{O}_X$-module.

A *deformation* (M_S, S, s) of M is a coherent $\mathcal{O}_{X\times S}$-module M_S which is S-flat where S is of finite type over k and $s \in S$ is a closed point (called the central point) such that the fiber over s is M.

A deformation (M_R, R, r) is called *versal* if any other deformation (M_S, S, s) may be induced from it in the following sense. There exist a diagram

$$\begin{tikzcd}
(S', s') \arrow[d, "\alpha"', "etale"] \arrow[dr, "\beta"] & \\
(S, s) \arrow[r, dashed] & (R, r)
\end{tikzcd}$$

such that $\alpha^* M_S \cong \beta^* M_R$.

The versal deformation of a module

Theorem

If M is locally free on the complement of the closed point, then there exists a versal deformation (M_R, R, r) which is unique up to non-unique isomorphism in the etale topology.

Definition

Let M and N be two reflexive modules on X, and let (M_R, R, r) be the versal deformation of M. Let $\mathrm{Loc}(N)$ be the set of k-points $t \in R(k)$ such that the pullback M_t of M_R to t is isomorphic to N.

Then M *locally deforms to* N, denoted $M \dashrightarrow N$, if the Zariski closure $\overline{\mathrm{Loc}}(N)$ strictly contains the central k-point r corresponding to M.

If, possibly after restricting to a Zariski open set in R containing r, $\overline{\mathrm{Loc}}(N) \setminus \{r\} = \mathrm{Loc}(N)$ (as sets of k-points), then $\overline{\mathrm{Loc}}(N)$ is called an *absolute minimal stratum* of R and the local deformation of M to N is called *minimal*.

Motivation

Motivation

Classification in algebraic geometry

- discrete invariants
- moduli spaces
- jump phenomena (less studied)

Example: Reflexive modules on rational double points

By the McKay correspondence, the isomorphism classes are given by the first Chern class. Still such a reflexive module M in general has non-trivial local deformations ("jumps") and considering the resulting partial order provides us with an enrichment of the classification.

Motivation

Why we work with the versal deformation

- local deformation relation
- structure on the classification
- new invariants
- new classes of singularities

This may contribute to both representation theory and singularity theory.

The deformation graph

The deformation graph

Definition

Let $\mathbf{G}^{\text{def}}(X, r)$ denote the directed graph with vertices corresponding to isomorphism classes for rank r reflexive modules and where there is an arrow between vertices corresponding to M and N if $M \dashrightarrow N$.

Remark

The $\mathbf{G}^{\text{def}}(X, r)$ are invariants of X.

Notation

- A divisor d is *nef* if $d = \sum n_i D_i$ with $n_i \in \{0, 1, 2, \dots\}$ for all i, where D_i are transversal divisors with $E_i \cdot D_i = 1$.
- If d and d' are *nef* divisors, let $d \dashrightarrow d'$ if $d' - d$ is effective and supported on $E = \sum E_i$.

The graph of nef divisors

Definitions

- Let the *graph of nef divisors* $\mathbf{G}^{\mathsf{nef}}(X)$ be the directed graph with the nef divisors as vertices and edges given by $d \dashrightarrow d'$.
- Let $\mathbf{G}^{\mathsf{nef}}(X, r)$ be the full directed sub-graph of $\mathbf{G}^{\mathsf{nef}}(X)$ with vertices given as divisors $\mathrm{c}_1(\widetilde{M})$ (this is always nef) for reflexive modules on X of rank r.

Remark

By a result of Wunram $\cup_{r \geqslant 1} \mathbf{G}^{\mathsf{nef}}(X, r) = \mathbf{G}^{\mathsf{nef}}(X)$.

A map of directed graphs

Theorem (Ishii)

Assume X is rational. If $M \dashrightarrow N$ then $\mathrm{c}_1(\widetilde{M}) \dashrightarrow \mathrm{c}_1(\widetilde{N})$,i.e. there is a map of directed graphs

$$\mathrm{c}_1 : \mathbf{G}^{\mathrm{def}}(X, r) \longrightarrow \mathbf{G}^{\mathrm{nef}}(X, r)$$

which is surjective on the vertices.

Remark

This follows from Ishii's existence result for canonical resolutions of the Chern class stratification in the versal deformation space.

A surjectivity conjecture

Let X be a quotient singularity. Then

$$\mathrm{c}_1 : \mathbf{G}^{\mathrm{def}}(X, r) \rightarrow \mathbf{G}^{\mathrm{nef}}(X, r)$$

is *full*.

Evidence for the surjectivity conjecture

Theorem (Ishii)

Let X be a rational double point. If M and N have rank r and $\mathrm{c}_1(\widetilde{M}) \dashrightarrow \mathrm{c}_1(\widetilde{N})$, then $M \dashrightarrow N$. I.e.

$$\mathbf{G}^{\mathsf{def}}(X, r) \cong \mathbf{G}^{\mathsf{nef}}(X, r)\,.$$

Hence the surjectivity conjecture is true for rational double points.

Remark

This result may be seen as an *enrichment* of the McKay correspondence; in addition to the bijection on vertices, we have bijection on the arrows.

Proposition

If X is the cone over a rational normal curve, then the surjectivity conjecture holds, but $\mathbf{G}^{\mathsf{def}}(X, r) \ncong \mathbf{G}^{\mathsf{nef}}(X, r)$.

Main results

Cones over rational normal curves

Definition

Let $X = X_m$ be the affine cone over the m-uple embedding of $\mathbb{P}^1_k$ in $\mathbb{P}^m_k$, i.e. $\mathcal{O}_X = k[u^m, u^{m-1}v, \ldots, v^m]^{\mathrm{h}}$ where "h" denotes Henselisation.

Facts

- X_m has multiplicity m.
- The exceptional divisor $E \cong \mathbb{P}^1_k$. $E \sim -mD$ where D is any curve intersecting E transversally in one point, in particular $E^2 = -m$.
- $\operatorname{Pic} \widetilde{X} \cong \mathbb{Z}$ generated by D.
- The indecomposable reflexive $\mathcal{O}_X$-modules are $M_i = (u^i, u^{i-1}v, u^{i-2}v^2, \ldots, v^i)$ for $0 \leqslant i \leqslant m-1$ with $\widetilde{M}_i \cong \mathcal{O}_{\widetilde{X}}(iD)$ and they all have rank one.

The number of components of the versal deformation space

Theorem

Let X be the cone over a rational normal curve, and let M be a (not necessarily indecomposable) reflexive module on X. Then the number of irreducible components of the versal deformation space is

$$1 + \min\left(\left\lfloor \frac{d}{m} \right\rfloor, n\right) - \max\left(\left\lceil \frac{d+2r}{m} \right\rceil - r, 0\right)$$

where $m = \mathrm{mult}(X) \geq 3$, $r = \mathrm{rk}(M)$, $d = c_1(\widetilde{M})$ and $n < r$ is the multiplicity of the rank one indecomposable module with Chern class $m-1$ in M.

If $m > r$ and $n = r$ then there are $r-1$ components.

Structure of the versal deformation space

Theorem

Let X be the cone over the rational normal curve of degree m. Let M be any (not necessarily indecomposable) reflexive module on X. The minimal stratum in the versal deformation space of M is the *cone over a*

- *Segre embedding* times
- an *incidence variety* times
- an *affine space*

intersected with *certain hyperplanes and quadratic hyper surfaces.*

Structure of the versal deformation space

Theorem

Let X be the cone over the rational normal curve of degree m. Let M be any (not necessarily indecomposable) reflexive module on X. To each component of the versal deformation space of M, a resolution is given as the total space of a vector bundle on a Grassmannian. The vector bundle is a sum of copies of

- the cotangent bundle,
- the canonical sub-bundle,
- the dual of the canonical quotient bundle,
- and the trivial line bundle.

Via an embedding in a trivial bundle, we obtain the components by projection.

Other results

Theorem

Assume X is a rational surface singularity. The number of connected components $nc(X, r)$ in $\mathbf{G}^{\mathsf{def}}(X, r)$ satisfies

$$nc(X, r) \geqslant \#H$$

where $\#H$ denotes the number of elements in the group H. In the case X is a rational double point $nc(X, r) = \#H$.

Theorem

Let M be an indecomposable reflexive module on a rational double point. If $\operatorname{rk} M = 1$, then M is terminal. If $\operatorname{rk} M > 1$, then the possible deformations are classified.

The $\mathbf{D}_n$-case

$$\begin{tikzcd}
\delta_{2\left\lfloor \frac{n-2}{2} \right\rfloor} \arrow[d, dotted, no head] & \delta_{2\left\lceil \frac{n-2}{2} \right\rceil - 1} \arrow[d, dotted, no head] \\
\delta_4 \arrow[d, dashed, swap, "\mathbf{D}_{n-2}"] & \delta_5 \arrow[d, dashed, swap, "\mathbf{D}_{n-3}"] \\
\delta_2 \arrow[d, dashed, swap, "\mathbf{D}_n"] & \delta_3 \arrow[d, dashed, swap, "\mathbf{D}_{n-1}"] \\
\delta_0 & \delta_1
\end{tikzcd}$$

The $\mathbf{E}_6$-case

$$\begin{tikzcd}
\delta_4 \arrow[d, dashed, "\mathbf{D}_4"'] & \delta_5 \arrow[d, dashed, "\mathbf{D}_5"'] & \delta_3 \arrow[d, dashed, "\mathbf{D}_5"'] \\
\delta_1 + \delta_6 \arrow[d, dashed, "\mathbf{A}_5"'] & \delta_1 & \delta_6 \\
\delta_2 \arrow[d, dashed, "\mathbf{E}_6"'] & & \\
\delta_0 & &
\end{tikzcd}$$

The $\mathbf{E}_7$-case

$$\begin{tikzcd}
& \delta_4 \arrow[d, dashed, "\mathbf{D}_4"'] & & \delta_5 \arrow[d, dashed, "\mathbf{D}_5"'] \\
& \delta_1+\delta_6 \arrow[dl, dashed, "\mathbf{D}_6"'] \arrow[dr, dashed, "\mathbf{A}_5"] & & \delta_1+\delta_7 \arrow[d, dashed, "\mathbf{A}_6"'] \\
2\delta_1 \arrow[dr, dashed, "\mathbf{A}_1"'] & & \delta_2+\delta_7 \arrow[dl, dashed, "\mathbf{A}_5"'] \arrow[d, dashed, "\mathbf{E}_6"] & \delta_2 \arrow[d, dashed, "\mathbf{E}_6"'] \\
& \delta_3 \arrow[d, dashed, "\mathbf{D}_5"'] & 2\delta_7 \arrow[dl, dashed, "\mathbf{A}_1"] & \delta_7 \\
& \delta_6 \arrow[d, dashed, "\mathbf{D}_6"'] & & \\
& \delta_1 \arrow[d, dashed, "\mathbf{E}_7"'] & & \\
& \delta_0 & &
\end{tikzcd}$$

The $\mathbf{E}_8$-case

$$\begin{tikzcd}
 & \delta_4 \arrow[d, dashed, "\mathbf{D}_4"'] & & \\
 & \delta_1+\delta_6 \arrow[dl, dashed, "\mathbf{D}_6"'] \arrow[dr, dashed, "\mathbf{A}_5"] & & \\
2\delta_1+\delta_8 \arrow[dr, dashed, "\mathbf{A}_1"'] & & \delta_2+\delta_7 \arrow[dl, dashed, "\mathbf{A}_5"'] \arrow[dr, dashed, "\mathbf{E}_6"] & \\
 & \delta_3+\delta_8 \arrow[dl, dashed, "\mathbf{A}_6"'] \arrow[dr, dashed, "\mathbf{D}_5"'] & & 2\delta_7 \arrow[dl, dashed, "\mathbf{A}_1"'] \\
\delta_1+\delta_2 \arrow[dr, dashed, "\mathbf{A}_4"'] & & \delta_6+\delta_8 \arrow[dl, dashed, "\mathbf{A}_3"'] \arrow[d, dashed, "\mathbf{D}_6"] & \\
 & \delta_5 \arrow[d, dashed, "\mathbf{D}_5"'] & \delta_1+2\delta_8 \arrow[dl, dashed, "\mathbf{A}_1"] \arrow[dr, dashed, "\mathbf{E}_7"] & \\
 & ? & & ?
\end{tikzcd}$$

The $\mathbf{E}_8$-case, continued

$$\begin{tikzcd}
 & \delta_1+\delta_7 \arrow[dl, dashed, "\mathbf{D}_7"'] \arrow[dr, dashed, "\mathbf{A}_6"] & & 3\delta_8 \arrow[dd, dashed, "(*)"] \\
2\delta_1 \arrow[dr, dashed, "\mathbf{A}_1"'] & & \delta_2+\delta_8 \arrow[dl, dashed, "\mathbf{A}_6"] \arrow[dr, dashed, "\mathbf{E}_6"] & \\
 & \delta_3 \arrow[d, dashed, "\mathbf{D}_5"] & & \delta_7+\delta_8 \arrow[dll, dashed, "\mathbf{A}_2"] \\
 & \delta_6 \arrow[d, dashed, "\mathbf{D}_6"] & & \\
 & \delta_1+\delta_8 \arrow[dl, dashed, "\mathbf{E}_7"'] \arrow[dr, dashed, "\mathbf{A}_7"] & & \\
2\delta_8 \arrow[dr, dashed, "\mathbf{A}_1"'] & & \delta_2 \arrow[dl, dashed, "\mathbf{E}_6"] & \\
 & ? & &
\end{tikzcd}$$

The $\mathbf{E}_8$-case, continued

$$\begin{tikzcd}
\delta_7 \arrow[d, dashed, "\mathbf{D}_7"] \\
\delta_1 \arrow[d, dashed, "\mathbf{E}_7"] \\
\delta_8 \arrow[d, dashed, "\mathbf{E}_8"] \\
\delta_0
\end{tikzcd}$$

Examples

A first example

- $X = \operatorname{Spec} k[x,y,z]/(xy - z^2)$ with resolution $\pi : \widetilde{X} \to X$
- McKay correspondence $\implies \exists!$ non-trivial indecomposable M_1
- $\widetilde{M}_1 = \mathcal{O}_{\widetilde{X}}(D)$ and $M_1 = \pi_* \mathcal{O}_{\widetilde{X}}(D)$ where $D \cdot E = 1$
- $M := M_1^2$, infinitesimal deformations?
- $\operatorname{Ext}^1_X(M, M) \cong \operatorname{Ext}^1_{\widetilde{X}}(\widetilde{M}, \widetilde{M^\vee}^\vee) \cong \operatorname{Ext}^1_{\widetilde{X}}(\widetilde{M}, \widetilde{M}^\vee)$ by a lemma of Ishii
- We get

$$\begin{aligned}
\operatorname{Ext}^1_{\widetilde{X}}(\widetilde{M}, \widetilde{M}^\vee) &\cong \operatorname{Ext}^1_{\widetilde{X}}(\mathcal{O}_{\widetilde{X}}(D)^2, \mathcal{O}_{\widetilde{X}}(-D)^2)) \\
&\cong H^1(\widetilde{X}, \mathcal{H}om(\mathcal{O}_{\widetilde{X}}(D)^2, \mathcal{O}_{\widetilde{X}}(-D)^2)) \\
&\cong H^1(\mathbb{P}^1, \mathcal{H}om(\mathcal{O}^2, \mathcal{O}^2) \otimes \mathcal{O}^2(-2)) \\
&\cong \operatorname{End}(k^2) \otimes H^1(\mathbb{P}^1, \mathcal{O}(-2)) \\
&\cong \operatorname{End}(k^2)
\end{aligned}$$

Example continued

One can show that the versal deformation space is given as follows

$$R = \{\varphi \,|\, \varphi^2 = 0\} \subset \mathbb{A}^4 = \mathrm{End}(k^2)$$

It is an exercise to show that

$$\begin{aligned} R_{\mathrm{red}} &= \{\varphi \,|\, \det(\varphi) = 0, \mathrm{tr}(\varphi) = 0\} \\ &= \{\begin{pmatrix} z & x \\ y & -w \end{pmatrix} \,|\, zw - xy = 0, z - w = 0\} \qquad \cong X \end{aligned}$$

Let $V = k^2$ and consider

$$\begin{tikzcd} T = \{(\varphi, \Lambda) \,|\, \varphi(\Lambda) = 0, \varphi(V) \subseteq \Lambda\} \arrow[r, phantom, "\subset"] & \mathrm{End}(V) \times \mathbb{P}(V) \arrow[d, "\mathrm{pr}_1"'] \\ & \mathrm{End}(V) \end{tikzcd}$$

Example continued

The image of

$$\begin{tikzcd} T = \{(\varphi,\Lambda) \mid \varphi(\Lambda)=0, \varphi(V)\subseteq\Lambda\} \arrow[r, phantom, "\subset"] & \mathrm{End}(V)\times\mathbb{P}(V) \arrow[d, "\mathrm{pr}_1"'] \\ & \mathrm{End}(V) \end{tikzcd}$$

is in fact $R_{\mathrm{red}} \cong X$ and $T \cong \widetilde{X}$.

Moreover, as bundle over $\mathbb{P}^1$, T is the cotangent bundle.

Example

Consider $X = X_m$ with $m = 3$. There are two non-trivial indecomposable modules M_1 and M_2. We write (n_0, n_1, n_2) for the module $\mathcal{O}_X^{n_0} \oplus M_1^{n_1} \oplus M_2^{n_2}$. Let $M = M_2^3$.

The relevant part of the deformation graph is:

$$\begin{tikzcd}
(0,0,3) \arrow[d, dashed] & & d=6 \\
(1,1,1) \arrow[r, dashed] \arrow[d, dashed] & (0,3,0) & d=3 \\
(3,0,0) & & d=0
\end{tikzcd}$$

From this we deduce that reduced versal deformation space R_{red} has two components R^3 and R^0 corresponding to the two terminal modules; M_1^3 with Chern number 3 and $\mathcal{O}_X^3$ with Chern number 0.

Example continued

We consider $T^3 = \mathbf{Spec}\,\mathrm{Sym}_{\mathcal{O}_{\mathbb{P}^2}}\left(T_{\mathbb{P}^2} \oplus T_{\mathbb{P}^2}\right)$ on $A = \mathrm{Grass}(1,3) \cong \mathbb{P}^2$. We can show that $E = \mathrm{Spec}\,\mathrm{Sym}_k\,\mathrm{Ext}^1_X(M,M)^*$ $= \mathrm{Spec}\,k[\{z_{ij}^{(n)}\}] = M_{3,3} \times M_{3,3}$ where $M_{3,3}$ denotes the affine space of 3×3-matrices. We may consider $T^3 \subseteq E \times \mathbb{P}^2(w_1, w_2, w_3)$ as given by the minors of

$$\begin{bmatrix} w_1 & z_{1j}^{(1)} \\ w_2 & z_{2j}^{(1)} \\ w_3 & z_{3j}^{(1)} \end{bmatrix} \text{ and of } \begin{bmatrix} w_1 & z_{1j}^{(2)} \\ w_2 & z_{2j}^{(2)} \\ w_3 & z_{3j}^{(2)} \end{bmatrix} \quad \text{for } 1 \leqslant j \leqslant 3$$

and the entries of

$$\begin{bmatrix} z_{11}^{(1)} & z_{12}^{(1)} & z_{13}^{(1)} \\ z_{21}^{(1)} & z_{22}^{(1)} & z_{23}^{(1)} \\ z_{31}^{(1)} & z_{32}^{(1)} & z_{33}^{(1)} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \text{ and of } \begin{bmatrix} z_{11}^{(2)} & z_{12}^{(2)} & z_{13}^{(2)} \\ z_{21}^{(2)} & z_{22}^{(2)} & z_{23}^{(2)} \\ z_{31}^{(2)} & z_{32}^{(2)} & z_{33}^{(2)} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

Example continued

The first set of conditions on the matrices (z_{ij}^n) is the same as saying that we may write

$$(z_{ij}^n) = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \begin{bmatrix} \gamma_1^{(n)} & \gamma_2^{(n)} & \gamma_3^{(n)} \end{bmatrix}.$$

Then the second set of conditions may then be written

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \begin{bmatrix} \gamma_1^{(n)} & \gamma_2^{(n)} & \gamma_3^{(n)} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = 0.$$

or

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} (\gamma_1^{(n)} w_1 + \gamma_2^{(n)} w_2 + \gamma_3^{(n)} w_3) = 0$$

which gives $z_{11}^{(n)} + z_{22}^{(n)} + z_{33}^{(n)}$, $n = 1, 2$.

Example continued

The image of the projection of T^3 to E is R^3 and is given as the cone over the image of the Segre embedding $\mathbb{P}^5 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}(M_{3\times 3} \times M_{3\times 3})$,

$$\left(\left(\begin{bmatrix} \gamma_1^{(1)} \\ \gamma_2^{(1)} \\ \gamma_3^{(1)} \end{bmatrix}, \begin{bmatrix} \gamma_1^{(2)} \\ \gamma_2^{(2)} \\ \gamma_3^{(2)} \end{bmatrix} \right), \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right) \longmapsto \left(\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \begin{bmatrix} \gamma_1^{(1)} \\ \gamma_2^{(1)} \\ \gamma_3^{(1)} \end{bmatrix}^t, \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \begin{bmatrix} \gamma_1^{(2)} \\ \gamma_2^{(2)} \\ \gamma_3^{(2)} \end{bmatrix}^t \right)$$

intersected with the two hyperplanes $z_{11}^{(n)} + z_{22}^{(n)} + z_{33}^{(n)}$, $n = 1, 2$.
We can show that $T^0 \cong T^3$, hence $R^0 \cong R^3$ where T^0 is a bundle over the dual projective plane $\mathrm{Grass}(2,3) = \check{\mathbb{P}}^2$, and that the embedding of $T^0 \hookrightarrow E \times \check{\mathbb{P}}^2$ is given by the map above with the $(z_{ij}^{(n)})$-matrices transposed.
It follows that the intersection $R^3 \bigcap R^0$ is given by the image under the map above of the locus where the gamma vectors are parallel, which thus is 5-dimensional of codimension one in R. On the intersection the fiber of the versal family must be $N = \mathcal{O}_X \oplus M_1 \oplus M_2$ as N is the only reflexive module besides M which locally deforms to both the terminal modules.

Representing Ishii's functor

Ishii's functor

Definition

Let the functor

$$\mathcal{T}^d_{M_R} : \mathsf{Sch}_k \longrightarrow \mathsf{Sets}$$

for any S in Sch_k be given as the set $\mathcal{T}^d_{M_R}(S)$ of equivalence classes of triples $(\mathcal{E}_S, \psi_S, \varphi_S)$ where

i) $\mathcal{E}_S$ is a locally free sheaf on $\widetilde{X}_S = \widetilde{X} \times_X X_S$,

ii) $\mathrm{R}^1\pi_*\mathcal{E}_t = 0$ and $\mathrm{c}_1(\mathcal{E}_t) = d$ for all pullbacks $\mathcal{E}_t$ of $\mathcal{E}_S$ to k-points $t \in S(k)$,

iii) $\psi_S : S \to R$ makes S into an R-scheme.

iv) $\varphi_S : \pi_{S*}\mathcal{E}_S \xrightarrow{\simeq} \psi_S^* M_{R'}$ on $X_S = X_{R'} \times_{R'} S$.

Two triples $(\mathcal{E}_S, \psi_S, \varphi_S)$ and $(\mathcal{E}'_S, \psi'_S, \varphi'_S)$ are equivalent if $\psi_S = \psi'_S$ and there is an isomorphism $\tau : \mathcal{E}_S \xrightarrow{\simeq} \mathcal{E}'_S$ such that $\pi_{S*}(\tau) = (\varphi'_S)^{-1}\varphi_S$.

Ishii's theorem

Theorem

Let $R' = R_{\text{red}}$ be the reduced versal deformation space and let $M_{R'}$ be the induced family. The functor $\mathcal{T}^d_{M_{R'}}$ is represented by a R'-scheme $\psi_{T^d} : T^d = T^d_{M_{R'}} \to R'$ which is projective over R', regular, non-empty for a finite set of Chern classes d, and their images $\{R^d\}$ in R' constitutes a filtration of R'. There is an isomorphism between the locus S^d in R' of reflexive modules with first Chern class equal to d and the open set in $T^d_{M_{R'}}$ corresponding to full sheaves.

The fiber functor

Definition

Let $\mathcal{A} = \mathcal{A}_M^d$ be the sub-functor of $\mathcal{T}^d$ of tuples $(\mathcal{E}_S, \psi_S : S \to R, \varphi_S)$ where ψ_S factorises through $\operatorname{Spec} k$, i.e. is trivial.

Proposition

Let $d = \mathrm{c}_1(\widetilde{M}) + sE$ with $0 \leqslant s \leqslant r$ where r is the multiplicity of M_{m-1} in M. Then the functor $\mathcal{A}_M^d$ is represented by the Grassmannian $A = \mathrm{Grass}(s, r)$.

About the proof

Assume $(\mathcal{E}, \varphi)$ represent an element in $\mathcal{A}_M^d(k)$. We get $\widetilde{M} \subseteq_\varphi \mathcal{E} \subseteq_\varphi \widetilde{M^\omega}^\omega$. Dividing out by $\widetilde{M}$ we get $\overline{\mathcal{E}} \subseteq \widetilde{M^\omega}^\omega / \widetilde{M} \cong \mathcal{O}_E(-1)^r$. We get $\overline{\mathcal{E}} \cong \mathcal{O}_E(-1)^s$. Twisting and taking global sections we get $k^s \subset k^r$. This sets up the bijection on points.

The fiber functor

Definition

Assume $\mathcal{M}$ is a quasi-coherent sheaf on $\widetilde{X} \times S$. Let $\mathcal{E}xt^{*}_{\widetilde{X} \times S/S}(\mathcal{M}, -)$ denote the derived functor of $q_* \mathcal{H}om_{\widetilde{X} \times S}(\mathcal{M}, -) : \mathrm{Mod}_{\widetilde{X} \times S} \to \mathrm{Mod}_S$.

Proposition

Let $\mathcal{E}_A$ be the universal sheaf on $A = \mathrm{Grass}(s, r)$ representing $\mathcal{A} = \mathcal{A}^d_M$. There is a natural injective map

$$\mathcal{E}xt^1_{\widetilde{X} \times A/A}(\mathcal{E}_A, \mathcal{E}_A) \hookrightarrow \mathrm{Ext}^1_X(M, M) \times A.$$

Remark

Note that on a closed point $a \in A$ this map specialises to

$$\mathrm{Ext}^1_{\widetilde{X}}(\mathcal{E}_a, \mathcal{E}_a) \hookrightarrow \mathrm{Ext}^1_X(M, M).$$

Representing Ishii's functor

Theorem

The image of

$$\mathcal{E}xt^1_{\tilde{X}\times A/A}(\mathcal{E}_A,\mathcal{E}_A)\hookrightarrow \mathrm{Ext}^1_X(M,M)\times A$$

represents the functor $\mathcal{T}^d_{M_R}$.