Foliations in Geometry and Arithmetic

X algebraic/analytic variety or manifold

(integrable) foliation

$$\mathcal{F} \subseteq T_X, \quad D, E \in \mathcal{F} \Rightarrow [D, E] \in \mathcal{F}$$

$$\forall x \in X\colon \mathcal{F}/\mathfrak{m}_x \hookrightarrow \mathrm{Hom}_{\mathbf{k}_x}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbf{k}_x)$$

Zariski, Lipman, Nagata argument
Locally:
$D_1, D_2, \ldots, D_k \in \mathcal{F}$,
$\mathcal{F} = \mathcal{O}D_1 + \mathcal{O}D_2 + \cdots + \mathcal{O}D_k$
$f_1, f_2, \ldots, f_k \in \mathcal{O}$,
$D_i(f_j) = \delta_{ij}$

$$[D_i, D_j] = \sum_k f_{ij}^k D_k$$

$$0 = [D_i, D_j]f_\ell = \sum_k f_{ij}^k D_k(f_\ell) = f_{ij}^\ell$$

$\rightsquigarrow [D_i, D_j] = 0$

Frobenius' theorem

Characteristic 0, algebraic version:

$$f \mapsto \sum_{\alpha} \frac{D^{\alpha}(f)}{\alpha!} t^{\alpha}$$

$\rightsquigarrow \widehat{\mathcal{O}}_x \cong R[[t_1, \ldots, t_k]]$

Positive characteristic p:

Add $D \in \mathcal{F} \Rightarrow D^p \in \mathcal{F}$

$\rightsquigarrow$

$[D_i, D_j] = 0 \wedge D_i^p = 0$

$\rightsquigarrow$

$\mathcal{O}_x \cong S[t_1, \ldots, t_k]/(t_1^p - f_1, \ldots, t_k^p - f_k)$

(Variant) special case:
Characteristic 0.
$\forall v \in T_{X,x} \exists D \in T_X \colon D_x = v \Rightarrow$
x smooth point

Applications to smoothness
of moduli problems:

Smoothness of moduli
of Calabi-Yau varietes
(one of many proofs)

In positive characteristic
conclusions are weaker

Formal vs algebraic integrability

Characteristic 0:

Foliation $\mathcal{F} \subseteq T_X \iff$
Formal maps $\hat{X}_x \to \hat{Y}$, $\mathcal{F} = T_{\hat{X}_x/\hat{Y}}$

Algebraically integrable $\iff$
(Birational) map $X \to Y$, $\mathcal{F} = T_{X/Y}$

Example:

$$\mathcal{F} = \mathbf{k}[x, y](x\partial/\partial x + \alpha y\partial/\partial y)$$

Constants $f(yx^{-\alpha})$,
algebraically integrable $\iff$ $\alpha \in \mathbf{Q}$

Necessary conditions for algebraic integrability

$X \to Y$ algebraic map (over $\mathbf{C}$ say)
"Arithmetic thickening"

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} \\ & \searrow \quad \swarrow & \\ & \mathbf{Spec}\, R & \end{array}$$

p prime $\Rightarrow T_{\mathcal{X}/\mathcal{Y}}/pT_{\mathcal{X}/\mathcal{Y}} \subseteq T_{\overline{\mathcal{X}}}$ *p-integrable*
i.e., $D \in \mathcal{F} \Rightarrow D^p \in \mathcal{F}$

Example:

$D = x\partial/\partial x + \alpha y\partial/\partial y \Rightarrow$

$D^p = x\partial/\partial x + \alpha^p y\partial/\partial y$

p-integrable $\iff$ $\alpha^p \equiv \alpha$ mod p

For infinitely many $p \Rightarrow \alpha$ algebraic (easy)

All but finitely many $p \Rightarrow$

$\alpha \in \mathbf{Q}$ (number theory)

Remark:

$\alpha = i$: $\alpha^p \equiv \alpha$ mod p $\iff$ $p \equiv 1$ mod 4

A conjecture on existence of compact leaves

Conjecture: *A foliation almost all of whose reductions are p-integrable is algebraically integrable.*

Remark: Over the complex numbers the leaves of the foliation always exist as analytic (not necessarily closed) subvarieties. It can be shown that algebraic integrability is equivalent to the closure of the leaves in some compactification of the base space still being analytic.

Example cases

Example:
$(X, \mathcal{E}, \nabla)$ an integrable connection

$$\nabla_{[D,E]} = [\nabla_D, \nabla_E]$$

Necessary for existence of horisontal sections:
$\forall D \in T_X\colon \nabla_D s = 0$

Characteristic p: p-integrability
$\forall D \in T_X\colon \nabla_D^p = \nabla_{D^p}$

Grothendieck:

Conjecture: *An integrable connection almost all of whose reductions are p-integrable has a full set of algebraic horisontal sections.*

Connections, cont'd

$\mathcal{E} = \mathbf{k}[x, x^{-1}]e$, $\nabla_{d/dx} e = \alpha x^{-1} e$
Horisontal section $x^{-\alpha} e$
algebraic $\iff \alpha \in \mathbf{Q}$

p-integrable:
$\nabla^p_{xd/dx} e = \alpha^p e$, $\nabla_{(xd/dx)^p} e = \alpha e$
p-integrable $\iff \alpha^p \equiv \alpha$ mod p

$\mathcal{E} = \mathbf{k}[x]v$, $\nabla_{d/dx} v = v$
Horisontal section $e^{-x} v$

p-integrable:
$\nabla^p_{d/dx} v = v$, $\nabla_{(d/dx)^p} e = 0$
Never p-integrable

Compact leaf conjecture

$\Rightarrow$

Grothendieck's conjecture

Associate to $(X, \mathcal{E})$ the frame bundle $\mathcal{F}(\mathcal{E}) \to X$. $\nabla \rightsquigarrow$ foliation on $\mathcal{F}(\mathcal{E})$

∇ integrable $\iff$ foliation integrable
∇ p-integrable $\iff$ foliation p-integrable

full set of horisontal sections on leaf of foliation

Compact leaf conjecture
$\Rightarrow$
Grothendieck's conjecture

Translation invariant foliations

A abelian variety (complex torus)
$\mathcal{F} \subseteq T_A$ *translation invariant* foliation
$\mathcal{F} \leftrightarrow \mathcal{F}_0 \subseteq T_{A,0} = \mathrm{Lie}(A)$

Commutator of translation invariant vector fields equals commutator of Lie algebra

A commutative $\rightsquigarrow$ Lie(A) commutative
$\rightsquigarrow$ $\mathcal{F}$ always integrable

Lie(A) has p'th power
p-integrability $\iff$ $D \in \mathcal{F}_0 \Rightarrow D^p \in \mathcal{F}_0$
Compact leaf exists $\iff$
$\mathcal{F}_0 = \mathrm{Lie}(B)$, $B \subseteq A$ abelian subvariety

Remark: The corresponding problem for an affine group is essentially trivial.

Elliptic curves

E, E' elliptic curves (over $\mathbf{Q}$ for simplicity)
$V \subset \mathrm{Lie}(E \times E') = \mathrm{Lie}(E) \times \mathrm{Lie}(E')$,
$\dim V = 1$, $V \neq \mathrm{Lie}(E), \mathrm{Lie}(E') \Rightarrow$
V graph of linear iso $f\colon \mathrm{Lie}(E) \to \mathrm{Lie}(E')$

p-integrable $\iff$
f commutes with p'th power map F
$\mathrm{Tr}_E F \equiv \mathrm{Tr}_{E'} F$ mod p
(mod p trace formula)

Hasse: $|\mathrm{Tr}_E F|, |\mathrm{Tr}_{E'} F| \leq 2\sqrt{p} \rightsquigarrow$
$\mathrm{Tr}_E F = \mathrm{Tr}_{E'} F$ (p large) $\Rightarrow$ (Faltings)
E and E' isogenous $\rightsquigarrow$ assume $E = E'$
$\Rightarrow f$ mult by α, $\rightsquigarrow$
$F\colon \mathrm{Lie}(E) \to \mathrm{Lie}(E)$ non-zero $\Rightarrow$
$\alpha^p \equiv \alpha$ mod p

Case 1: E has complex multiplication
$F\colon \text{Lie}(E) \to \text{Lie}(E)$ non-zero $\iff$
p splits in field K
of complex multiplications
$\rightsquigarrow \alpha \in K \Rightarrow f$ graph of isogeny

Case 2: E no complex multiplications
$F\colon \text{Lie}(E) \to \text{Lie}(E)$ non-zero $\iff$
E has *ordinary* reduction
true for set of primes of density 1 $\Rightarrow$
$\alpha \in \mathbf{Q} \Rightarrow f$ graph of isogeny

Remark:
Similar arguments work in some other cases, notably for all abelian varieties with complex multiplication.

Evidence/inspiration for the conjecture.

Mori/Miyaoka construction of rat'l curves:

First idea (Mori):
Under positivity assumptions on tangent bundle construct rational curves in positive characteristic (use *Frobenius map*) of bounded degree and conclude characteristic 0 existence by finiteness of Hilbert schemes.

Second idea (Miyaoka):
Use only positivity assumptions on subbundle.
These positivity assumptions imply integrability and p-integrability of subbundle.
The rational curves are leaves.

Major positive result

Bost's theorem:

Theorem *If there is an analytic map* $\mathbf{C}^n \to X$ *which is generically a submersion, then the compact leaf conjecture is true for any foliation on* X.

Special case:

Theorem *Every translation invariant foliation on an abelian variety almost all of whose reductions are* p*-integrable corresponds to an abelian subvariety.*

Scattered examples

Discrete cofinite subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbf{R}) \times \mathrm{SL}_2(\mathbf{R})$
irreducible:
$(\Gamma \cap \mathrm{SL}_2(\mathbf{R}) \times \{1\})(\Gamma \cap \{1\} \times \mathrm{SL}_2(\mathbf{R}))$
not of finite index in Γ

$X = \mathbf{H} \times \mathbf{H}/\Gamma$ algebraic surface
$\mathbf{H} \times \{z\}$ and $\{z\} \times \mathbf{H}$ give foliations
Γ irreducible $\Rightarrow$ not alg. integrable
Conjecture $\Rightarrow$
not p-integrable for ∞ many p.

Remark: all $\mathbf{C}^n \to X$ are constant

Example:
K real quadratic field
X Hilbert modular surface
p-integrable $\iff$ p splits in K

$\mathcal{A}_g$ moduli space of principally polarised abelian varieties ($g > 1$)
$\mathcal{X}_g \to \mathcal{A}_g$ the universal abelian variety
$\mathbf{P} := \mathbf{P}(\mathrm{Lie}(\mathcal{X}_g/\mathcal{A}_g)) \to \mathcal{A}_g$,
the bundle of lines in $\mathrm{Lie}(\mathcal{X}_g/\mathcal{A}_g)$

Tautological $\mathcal{O}(-1) \subset \mathrm{Lie}(\mathcal{X}_g/\mathcal{A}_g)_{\mathbf{P}} \rightsquigarrow$
translation invariant foliation

$$\mathcal{F} \subset T_{\mathbf{P}\times_{\mathcal{A}_g}\mathcal{X}_g}$$

compact leaves $\leftrightarrow$
sub-elliptic curves of $\mathcal{X}_g$

compact leaves are *dense*
but *not* algebraically integrable

Paradox

Vanishing of p-curvature $\psi\colon \mathcal{F} \to T_X/\mathcal{F}$,
$\psi(D) := D^p \bmod \mathcal{F}$ is *closed* condition
Contradiction?

All compact leaves reduce to proper subvariety!