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On Metrics with Harmonic Curvature and non Trivial Ricci Tensor

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1. Yang-Mills connections and metrics with harmonic curvature

Let (M, g) be a compact n-dimensional C^∞ Riemannian manifold with $n \geq 3$,
∇ is its Levi-Civita connection
$\mathcal{R}$ is its curvature tensor,
r its Ricci tensor
W its Weyl conformal tensor
R its scalar curvature.
d is the exterior differentiation
δ is its divergence or its formal adjoint, viewed as differential forms on M.

According to the second Bianchi identity :

$$\delta\mathcal{R} = -dr \qquad i.e. \quad \nabla^i\mathcal{R}_{ijkl} = \nabla_k r_{hj} - \nabla_j r_{hk}$$

The sign conventions are such that
$r_{ij} = \mathcal{R}^l_{ilj}, \quad R = g^{ij} r_{ij}.$

$$(n-2)\delta W = -(n-3)d[r - \frac{Rg}{2n-2}].$$

We say (M, g) has harmonic curvature if

$$\delta\mathcal{R} = 0$$

which is called Yang-Mills equation.
That is the Euler-Lagrange equation of the Yang-Mills functional

$$\mathcal{YM}(D) = \frac{1}{2}\int_M ||\mathcal{R}^D||dv,$$

on the space $\mathcal{C}_E$ of connections in the vector bundle $E \to M$, where $\mathcal{R}^D$ is the curvature associated to the connection $D \in \mathcal{C}_E$.
So, the Levi-Civita connection ∇ is a Yang-Mills connection on the tangent bundle of M. In this way, ∇ is a critical point of the Yang-Mills functional.

More generally, a symmetric $(0, 2)$ tensor field C on (M, g) is called Codazzi tensor if it verifies the Codazzi equation $dC = 0$ i.e.

$$\nabla_j C_{ik} = \nabla_k C_{ij}.$$

We say C is a non trivial tensor if $\nabla_k C_{ij} \neq 0$.

Classical properties of metrics with harmonic curvature are summarized in the following lemma, A. Besse [B] chap. 16

Lemma 1 *Let* (M, g) *be a Riemannian manifold of dimension* $n \geq 3$*. The following holds*
(1) if $n = 3$, (M, g) *has harmonic curvature* $\delta\mathcal{R} = 0$ *if and only if it is conformally flat (*$W \equiv 0$*) and has constant scalar curvature* $R = Cte$*.*
(2) If $n \geq 4$, $\delta W = 0$ *and* M *has constant scalar curvature, then* $\delta\mathcal{R} = 0$ *.*
(3) (M, g) *has harmonic curvature if and only if its Ricci tensor is a Codazzi tensor (i.e.* $dr = 0$*).*
(4) If (M, g) *is a Riemannian product, then it has harmonic curvature if and only if any factor manifolds has harmonic curvature.*

2. Examples of manifolds with harmonic curvature

The following have harmonic curvatures
and trivial Ricci tensors:

- Local product of Einstein manifolds

- conformally flat manifolds with constant scalar curvature

- connected sums $Y = \# X_i$ where X_i are manifolds with negative constant scalar curvature

- warped product over Riemann surfaces

Notice that harmonicity condition is in a way a generalisation of the Einstein condition :

$$r - \frac{R}{n} g = 0 \qquad i.e. \qquad r_{ij} = \frac{R}{n} g_{ij}$$

In particular, this fact shows that every Einstein metric must have a trivial Ricci tensor $dr = 0$.

We may find many other examples of such manifolds in the chapter 16 of A. Besse [B].

Turn out now to non trivial cases of the Ricci tensor.

A. Derdzinski has given examples of metrics with harmonic curvature $dr \neq 0$.

3. Manifolds with harmonic curvature and non trivial Ricci tensor

Let (M_n, g), $n \geq 3$, be a compact C^∞ manifold with harmonic curvature. Suppose r is non trivial and has two distinct eigenvalues at each point. Then ([D] and [L]) (M, g) is covered isometrically by a manifold

$$(S^1 \times N, dt^2 + h^{4/n}(t) g_0),$$

where :
(i) S^1 is a circle of length T
(ii) (N, g_0) is a (n-1)-dimensional Einstein manifold with constant scalar curvature $R > 0$
(iii) h is a positive periodic function verifying the ODE

$$h'' - \frac{nR}{4(n-1)} h^{1-4/n} = -\frac{n}{4} C h \qquad (1)$$

for some constant $C > 0$.

Remarks

a) - h must be non constant, otherwise the corresponding metric has a trivial Ricci tensor.

b) - Notice that the manifolds $S^1 \times N$ are not conformally flat, unless (N, g_0) has constant sectional curvature.

c) - [L] and [K] proved if $T = 2k\pi\sqrt{\frac{n-1}{R}}$ where k is an enteger, then equation

$$Hess(f) - (\Delta f)g - fr = 0$$

has non trivial solutions. This equation generalizes the Obata equation $Hessf = -\frac{R}{n(n-1)}fg$ which has non trivial solutions only for the standard sphere S^n.

d) - For $n = 4$, Equation (1) becomes linear: $h'' + Ch - \frac{R}{3} = 0$.
So, it admits a 1-parameter family of non constant periodic solutions whenever $T = \frac{2k\pi}{\sqrt{C}}$. Thus, the corresponding 4-manifolds $S^1 \times N$ have harmonic curvatures and non trivial Ricci tensors.

4. Existence conditions for Derdzinski metrics

Theorem 2 *Let* (N, g_0) *be a (n-1)-dimensional Einstein manifold with positive scalar curvature* R, $n \neq 4$ *and* S^1 *be a circle of length* T *endowed with its standard metric.*
Consider the product $(S^1(T)\times N, dt^2 + h^{4/n}(t)g_0)$ *where* $h(t)$ *verifies Equation (1)*

$$h'' - \frac{nR}{4(n-1)}h^{1-4/n} = -\frac{n}{4}Ch \qquad for \quad C > 0.$$

Then, if T *satisfies the following inequalities*

$$2\pi\frac{(k-1)}{\sqrt{C}} < T \leq 2\pi\frac{k}{\sqrt{C}} \ , \qquad k \text{ is an integer} > 1,$$

there exist at least k *rotationnally invariant warped metrics* $dt^2 + h^{4/n}(t)g_0$ *on* $S^1(T) \times N$. *Moreover, these metrics have harmonic curvatures.*
Their Ricci tensors are non trivial only if $T > \frac{2\pi}{\sqrt{C}}$.
Conversely, if the warped metric of the type $dt^2 + h^{4/n}(t)g_0$ *on the manifold* $S^1(T) \times N$ *has harmonic curvature, then the function* $h(t)$ *verifies the ODE (1).*

5. Link with pseudo-cylindric metrics

Let the Riemannian cylindric product $(S^1 \times S^{n-1}, dt^2 + d\xi^2)$, where S^1 is the circle of length T and $(S^{n-1}, d\xi^2)$ is the standard sphere. Such a metric has a trivial Ricci tensor.

Moreover, we know that the number of Yamabe metrics is finite in the conformal class of the cylindric metric $[dt^2 + d\xi^2]$.

- g_c is a Yamabe metric on a n-dimensional Riemannian manifold (M, g) if there is a C^∞ function u_c such that the metric $g_c = u_c^{\frac{4}{n-2}} g$ has a constant scalar curvature -.

There is a conformal diffeomorphism between $S^n - \{p_1, p_2\}$ and $(S^1 \times S^{n-1}, dt^2 + d\xi^2)$, where S^1 is the circle of length T.

The non trivial Yamabe metrics on $S^1 \times S^{n-1}$, are called *pseudo-cylindric metrics.*
There are metrics of the form

$$g = u^{\frac{4}{n-1}}(dt^2 + d\xi^2)$$

where the C^∞ function u is a non constant positive solution of the Yamabe equation.

It has been shown (using an Alexandrov reflection argument) that any solution of

$$4\frac{n-1}{n-2}\Delta_{g_0}u + R_{g_0}u - R_g u^{\frac{n+2}{n-2}} = 0, \qquad (2)$$

is in fact a spherically symmetric radial function (depending on geodesic distance from either p or $-p$). Any solution of Equation (2) which yields a complete metric on the cylinder $IR \times S^{n-1}$ is of the form $u(t,\xi) = u(t)$, where $t \in IR$ and $\xi \in S^{n-1}$.

The background metric on the cylinder is the product $g_0 = dt^2 + d\xi^2$.

Therefore, the partial differential equation (2) is reduced to an ODE.
The cylinder has scalar curvature
$R(g_0) = (n-1)(n-2)$ and
$R(u^{\frac{4}{n-2}} g_0) = n(n-1)$. Thus $u = u(t)$ satisfies

$$\frac{d^2}{dt^2}u - \frac{(n-2)^2}{4}u + \frac{n(n-2)}{4}u^{\frac{n+2}{n-2}} = 0. \qquad (3)$$

It follows that a pseudo-cylindric metric on $(S^1(T) \times S^{n-1}, g_0)$ corresponds to a T-periodic positive solution of (3) and conversely.

Analysis of this equation shows us, that it has only one center $(\beta, 0)$ corresponding to the (trivial) constant solution

$$\beta = (\frac{n-2}{n})^{\frac{n-2}{4}},$$

We proved the following,

Theorem 3 *Consider the product manifold* $(S^1(T) \times S^{n-1}, g_0)$ *where* S^1 *is the circle of length* T *and* S^{n-1} *is the standard sphere.*

Under the condition

$$T(c) > T_1 = \frac{2\pi}{\sqrt{n-2}},$$

the Riemannian curvatures of the corresponding pseudo-cylindric metrics $g_c = u_c^{\frac{4}{n-2}} g_0$ *are harmonic and their Ricci tensors are non trivial.*

Moreover, any pseudo-cylindric metric may be identified to a Derdzinski metric up to a conformal transformation.