Hamiltonian systems:

What is integrability?

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EUROPA
1850
1891
С. В. КОВАЛЕВСКАЯ
1500 РУБ.
ROSSIJA · 1996
РОССИЯ

A classical problem

SUR LE PROBLÈME DE LA ROTATION

D'UN CORPS SOLIDE AUTOUR D'UN POINT FIXE[1]

PAR

SOPHIE KOWALEVSKI

À STOCKHOLM.

§ 1.

Le problème de la rotation d'un corps solide pesant autour d'un point fixe peut se ramener, comme on sait, à l'intégration du système d'équations différentielles suivant:

$$A\frac{dp}{dt} = (B - C)qr + Mg(y_0\gamma'' - z_0\gamma'), \qquad \frac{d\gamma}{dt} = r\gamma' - q\gamma'',$$

$$(1) \qquad B\frac{dq}{dt} = (C - A)rp + Mg(z_0\gamma - x_0\gamma''), \qquad \frac{d\gamma'}{dt} = p\gamma'' - r\gamma,$$

$$C\frac{dr}{dt} = (A - B)pq + Mg(x_0\gamma' - y_0\gamma), \qquad \frac{d\gamma''}{dt} = q\gamma - p\gamma'.$$

Les constantes $A, B, C, Mg, x_0, y_0, z_0$ qui figurent dans ces équations ont la signification suivante.

A, B, C sont les axes principaux de l'ellipsoïde d'inertie du corps considéré, relativement au point fixe.

M est la masse du corps;

g l'intensité de la force de gravité;

[1] Ce mémoire est le résumé d'un travail auquel l'Académie des Sciences de Paris, dans sa séance solennelle du 24 décembre 1888, a décerné le prix Bordin élevé de 3000 à 5000 francs.

Acta mathematica. 12. Imprimé le 22 janvier 1889.

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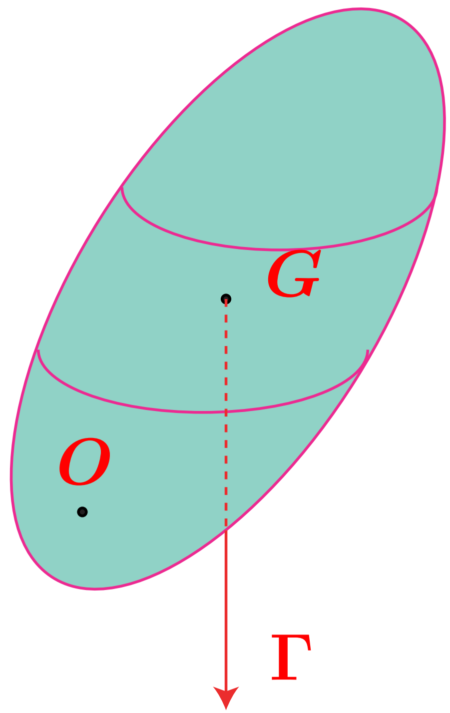

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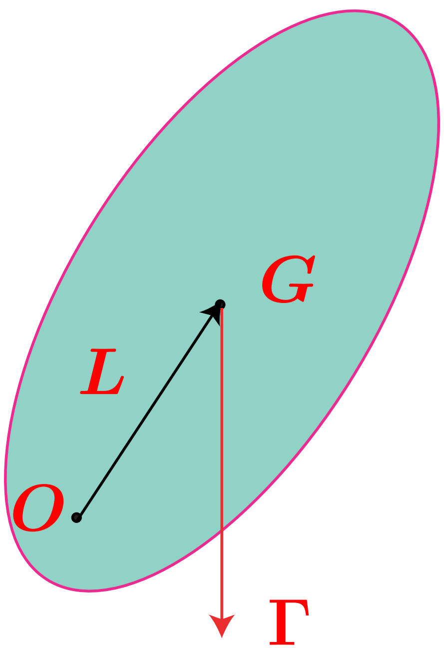

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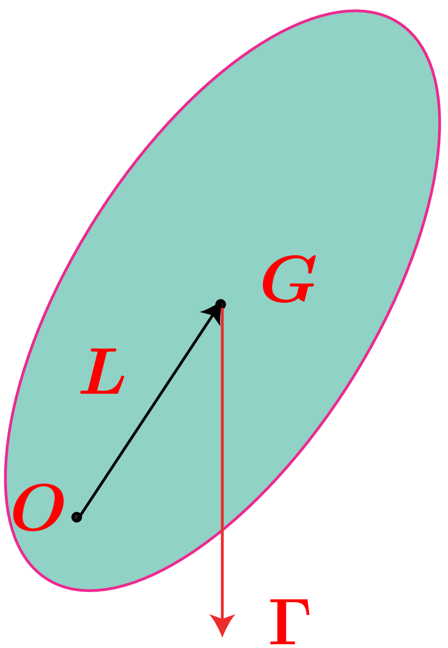

$$\boldsymbol{\Omega} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \boldsymbol{M} = \begin{pmatrix} Ap \\ Bq \\ Cr \end{pmatrix},$$

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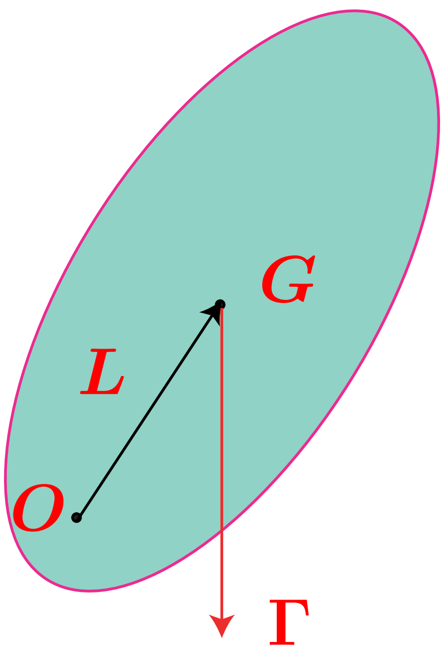

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$$A, B, C > 0, \quad \overrightarrow{OG} = L = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

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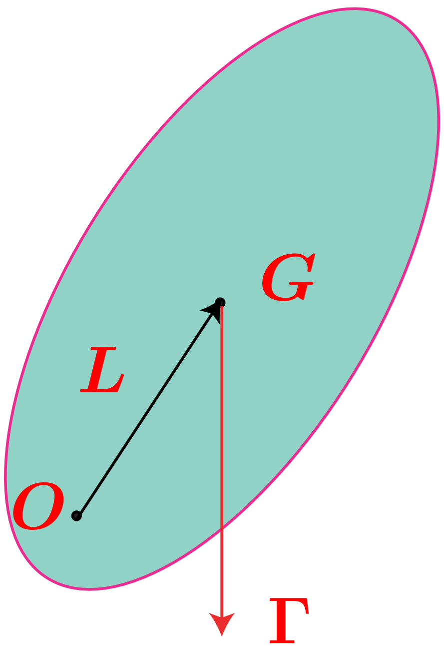

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\begin{cases}
\dot{M} = M \times \Omega + \Gamma \times L \\
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How to solve it?

x_0, y_0, z_0 les coordonnées du centre de gravité du corps considéré dans un système de coordonnées, dont l'origine est au point fixe et dont la direction coïncide avec celle des axes principaux de l'ellipsoïde d'inertie.

Jusqu'à présent on n'était parvenu à intégrer ces équations que dans deux cas particuliers:

1) Le cas de Poisson (ou d'Euler) où l'on a $x_0 = y_0 = z_0 = 0$,

2) Le cas de Lagrange où l'on a $A = B$, $x_0 = y_0 = 0$.

Dans ces deux cas l'intégration s'opère à l'aide des fonctions $\vartheta(u)$ dont l'argument est une fonction entière linéaire du temps.

Les six quantités $p, q, r, \gamma, \gamma', \gamma''$ sont dans ces deux cas des fonctions uniformes du temps, n'ayant d'autres singularités que des pôles pour toutes les valeurs finies de la variable.

Les intégrales des équations différentielles considérées conservent-elles cette propriété dans le cas général?

Si tel était le cas il faudrait pouvoir intégrer ces équations différentielles à l'aide de séries de la forme

$$
(2) \qquad \begin{aligned}
p &= t^{-n_1}(p_0 + p_1 t + p_2 t^2 + \dots),\\
q &= t^{-n_2}(q_0 + q_1 t + q_2 t^2 + \dots),\\
r &= t^{-n_3}(r_0 + r_1 t + r_2 t^2 + \dots),\\
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\gamma' &= t^{-m_2}(g_0 + g_1 t + g_2 t^2 + \dots),\\
\gamma'' &= t^{-m_3}(h_0 + h_1 t + h_2 t^2 + \dots),
\end{aligned}
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où $n_1, n_2, n_3, m_1, m_2, m_3$ désignent des nombres entiers positifs, et ces séries, pour pouvoir représenter le système général d'intégrales des équations différentielles considérées, devraient contenir *cinq* constantes arbitraires.

Il faut donc examiner si une pareille intégration est possible.

On s'assure facilement, en comparant les exposants des premiers termes dans les membres gauches et dans les membres droits des équations considérées que l'on doit avoir

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She requires the solutions to have this property

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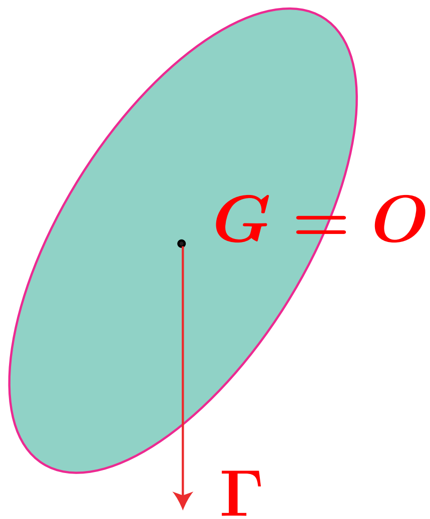

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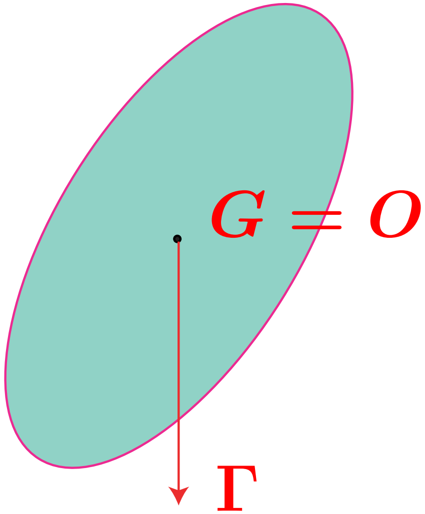

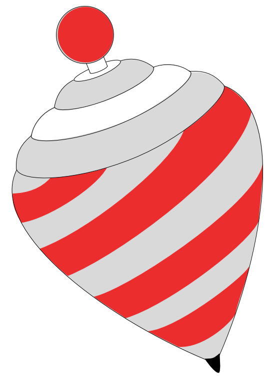

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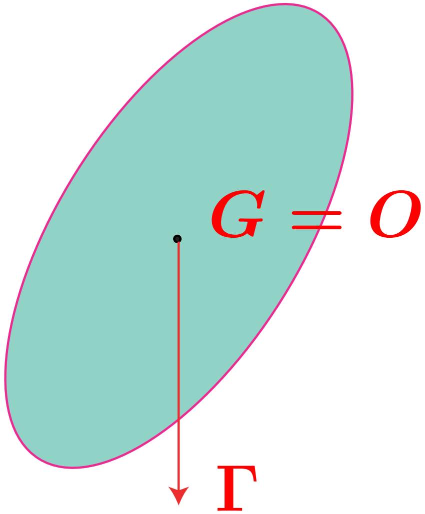

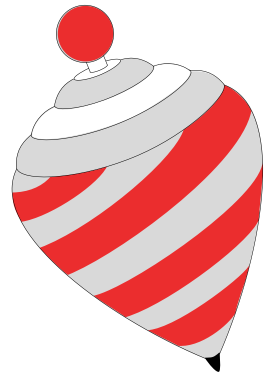

Plus a new case, $A = B = 2C$, $z_0 = 0$, the Kowalevski top.

By chance (?), she notices that the system

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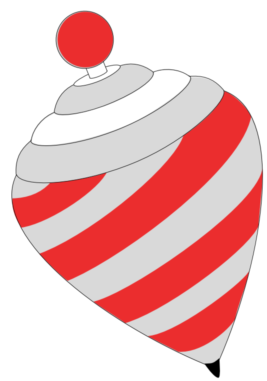

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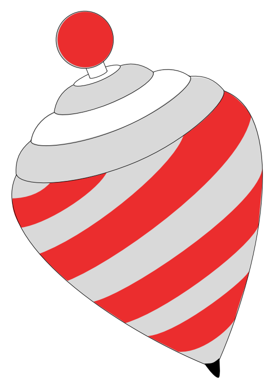

Then she solves the equations, using ϑ-functions on a genus-$\mathbf{2}$ curve.

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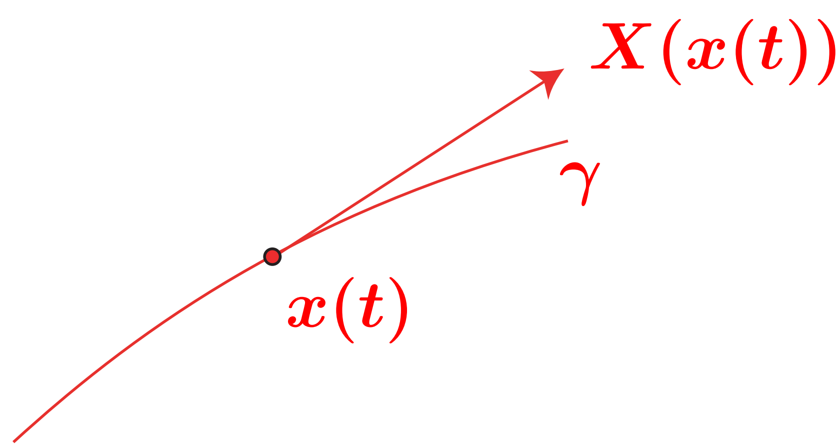

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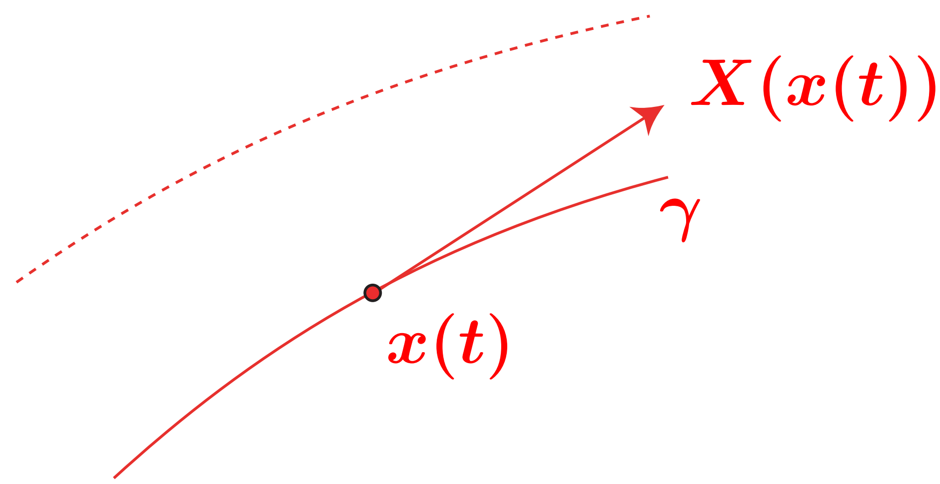

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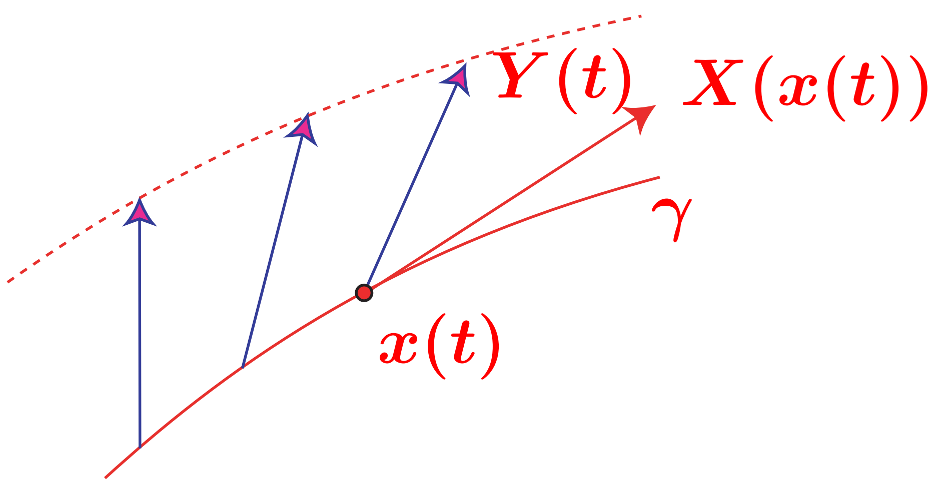

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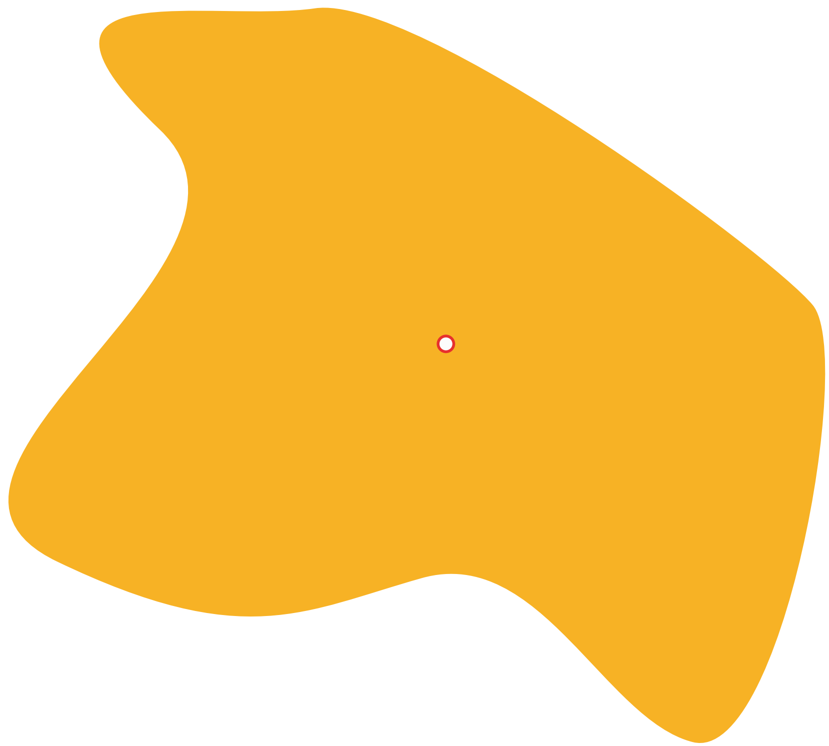
Applications to the integrability of geodesic flows on $SO(n)$, for instance.

Monodromy



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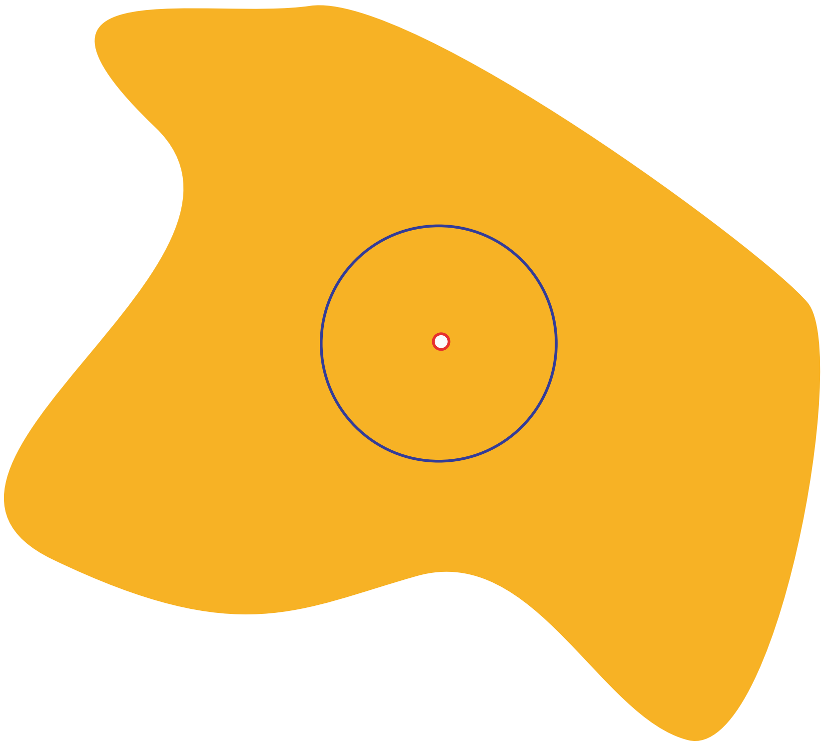

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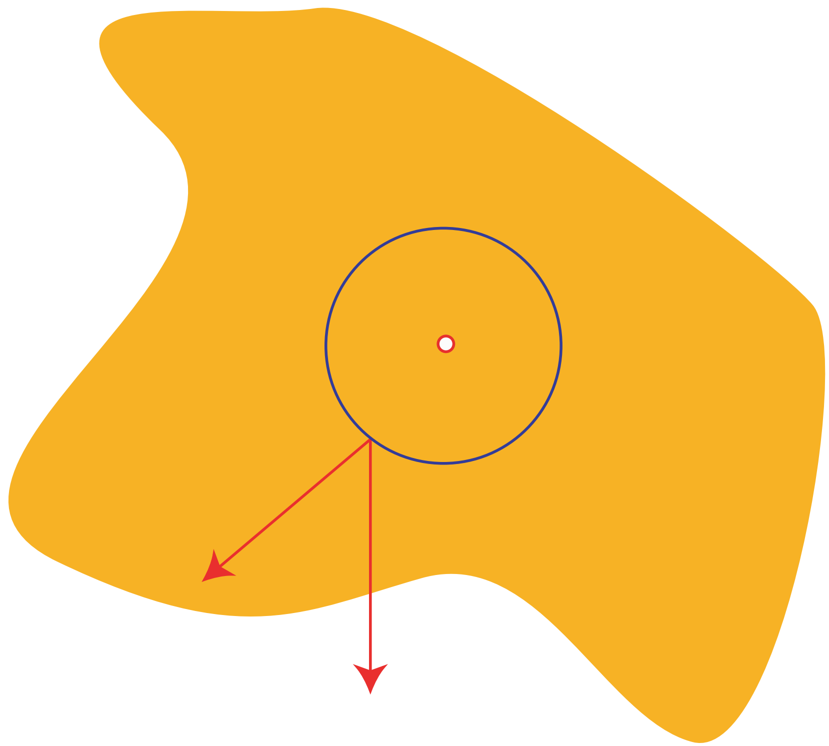

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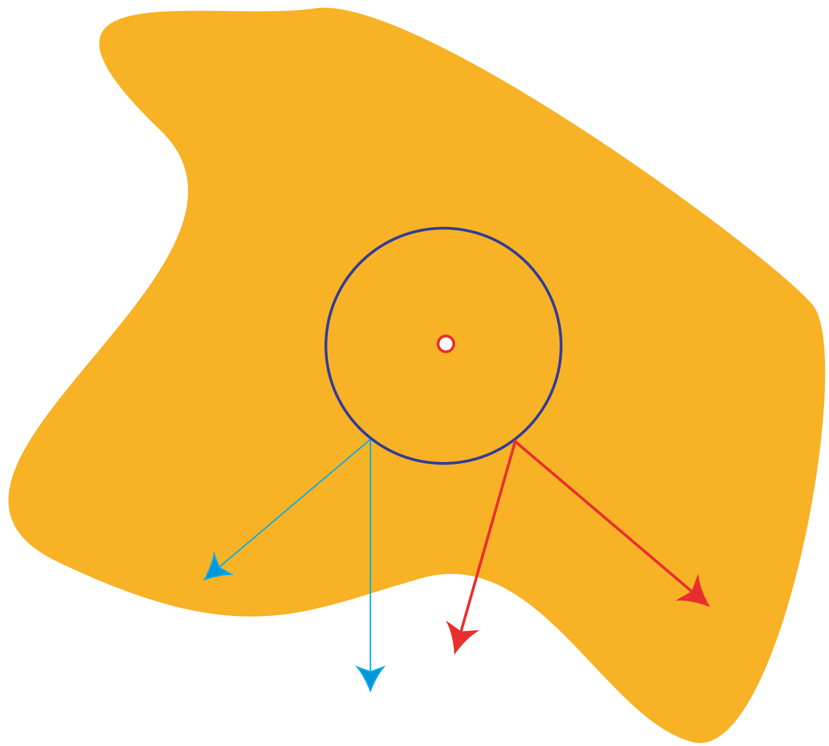

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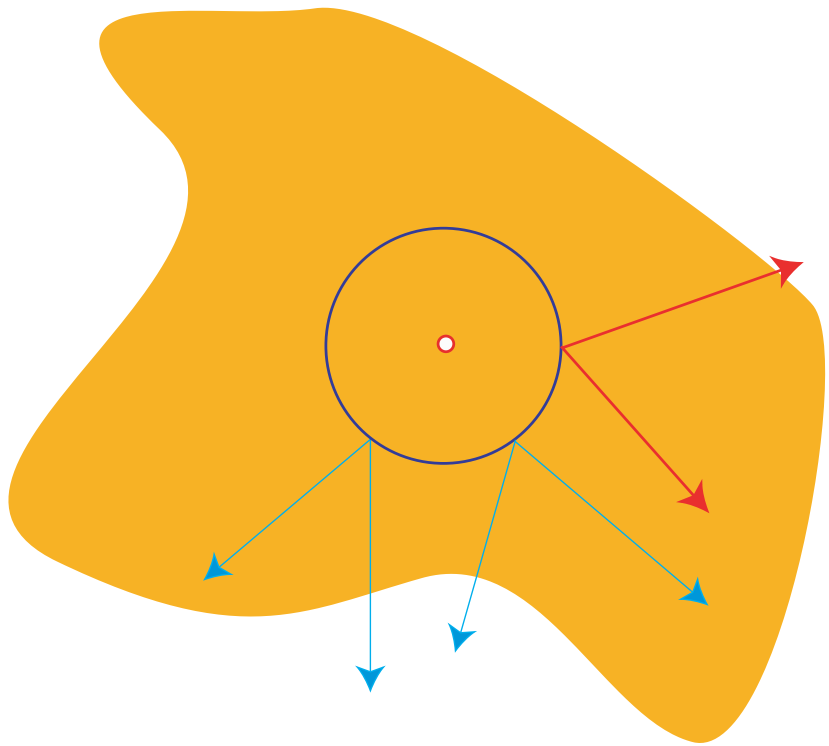

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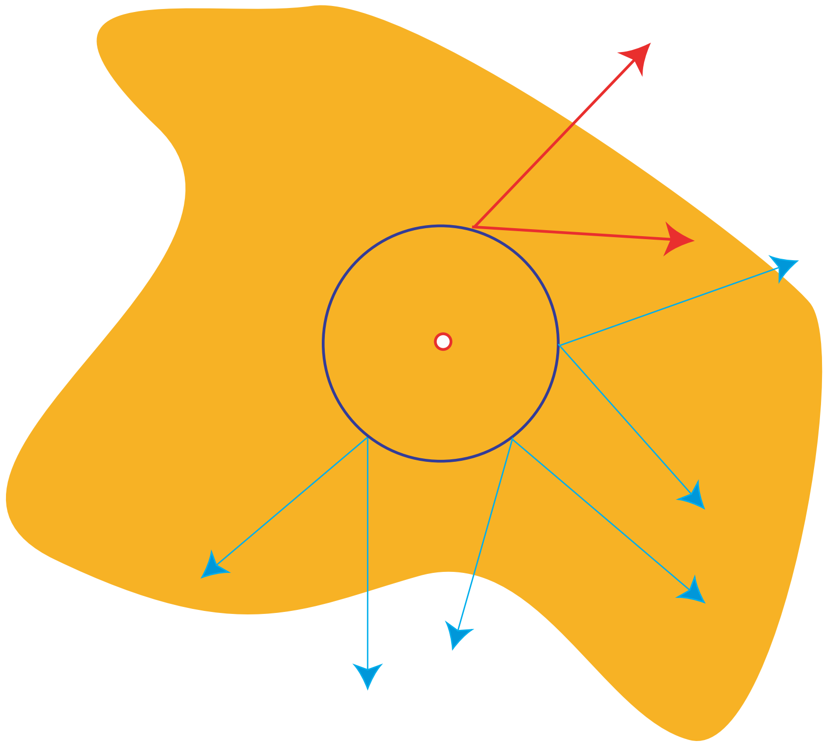

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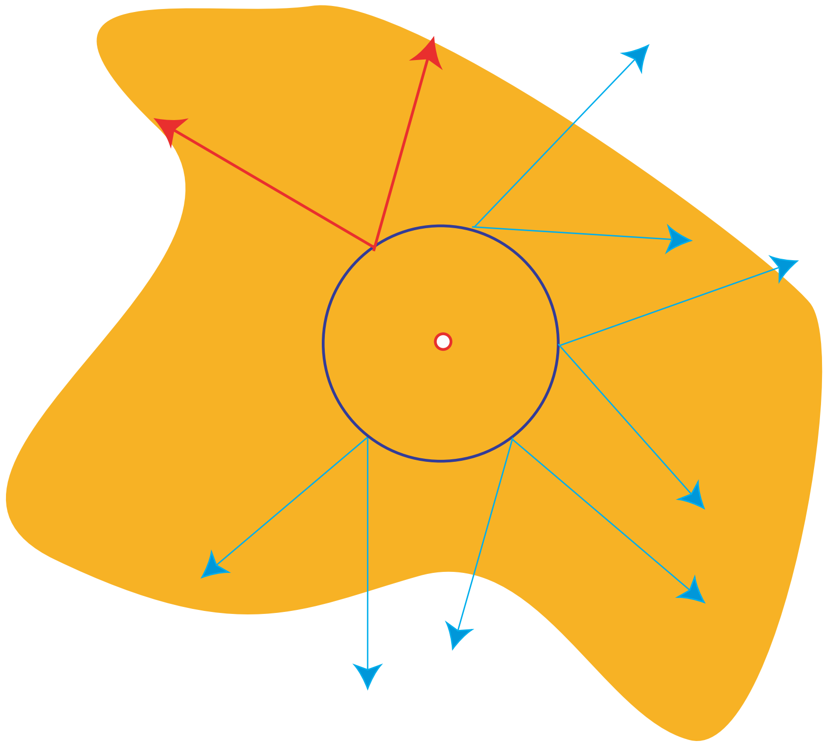

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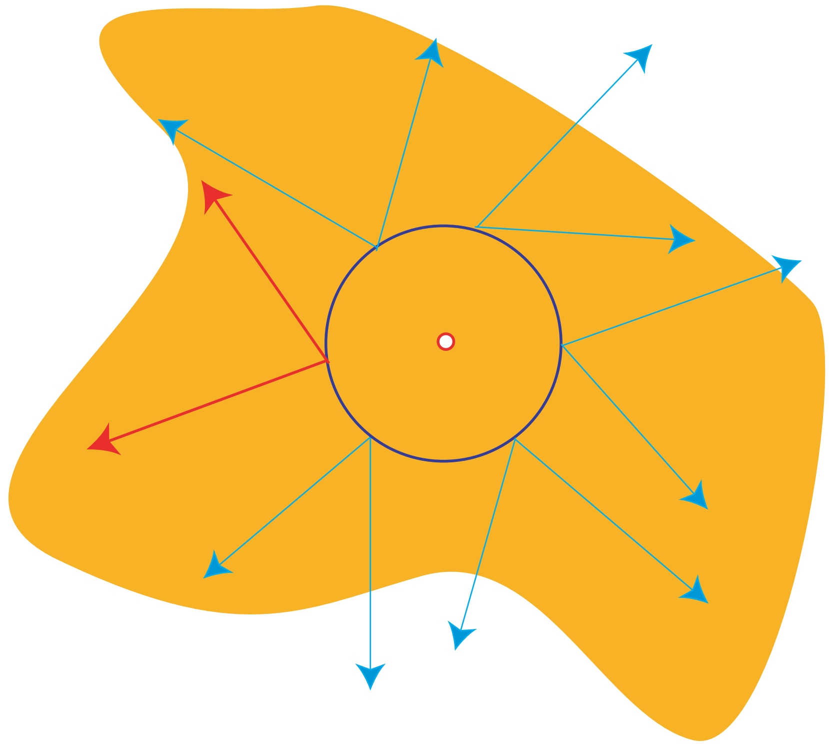

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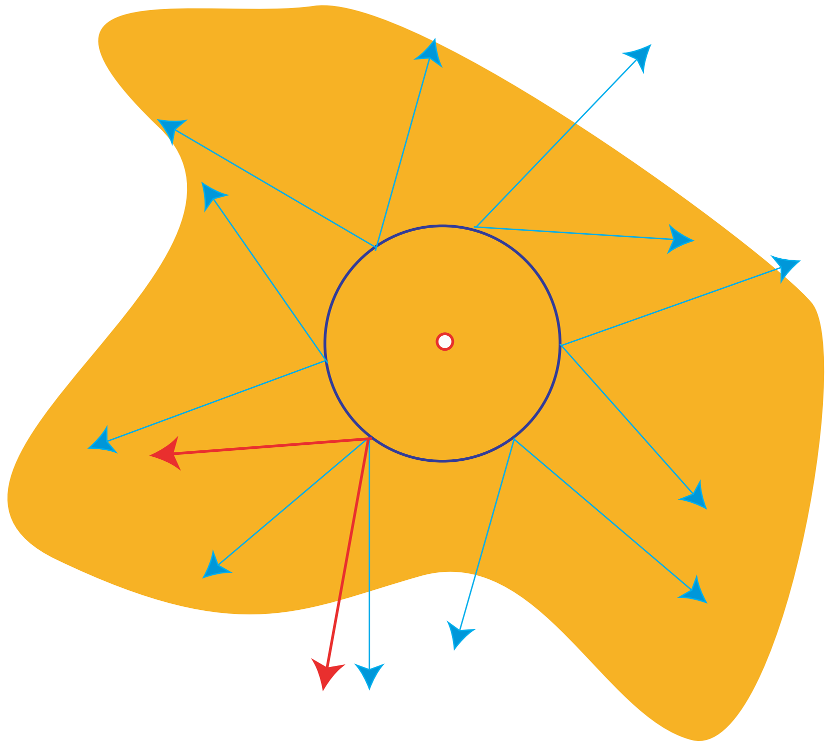

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Theorem (Morales-Ramis 1998). *If the Hamiltonian system is integrable (in the Liouville sense), the Galois group of the linearized system along any solution is virtually Abelian.*

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See for instance the EMS Newsletter, last january

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When the singularities of the linear differential equations are regular, the Galois group is just the Zariski closure of the monodromy group.

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In the problem of the rigid body, and in many other problems (geodesic flows on $SO(n)$,...), it turns out that the two properties give the same "integrable systems".

The case of the rigid body

Results:

- Kowalevskaya (1889): K. property $\Rightarrow$ either Euler-Poisson or Lagrange or Kowa. $\Rightarrow$ Liouville integrability.

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- Andrzej Maciejewski and Maria Przybylska (2004): same result, using Galois groups. And elliptic curves.

  Maciejewski (A.) and Przybylska (M.) — Differential Galois approach to the Non-Integrability of the Heavy Top Problem, *Ann. Fac. Sci. Toulouse*, 2004

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A\frac{dp}{dt} &= (B - C)qr + Mg(y_0\gamma'' - z_0\gamma'), & \frac{d\gamma}{dt} &= r\gamma' - q\gamma'', \\
B\frac{dq}{dt} &= (C - A)rp + Mg(z_0\gamma - x_0\gamma''), & \frac{d\gamma'}{dt} &= p\gamma'' - r\gamma, \\
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There is a special solution with $p = q = \gamma'' = 0$.
The differential system becomes

$$\frac{dr}{dt} = \gamma', \qquad \frac{d\gamma}{dt} = r\gamma', \qquad \frac{d\gamma'}{dt} = -r\gamma.$$

An elliptic curve

The solution is supported by the curve E_h

$$h = \frac{1}{2}r^2 - \gamma, \qquad \gamma^2 + \gamma'^2 = 1$$

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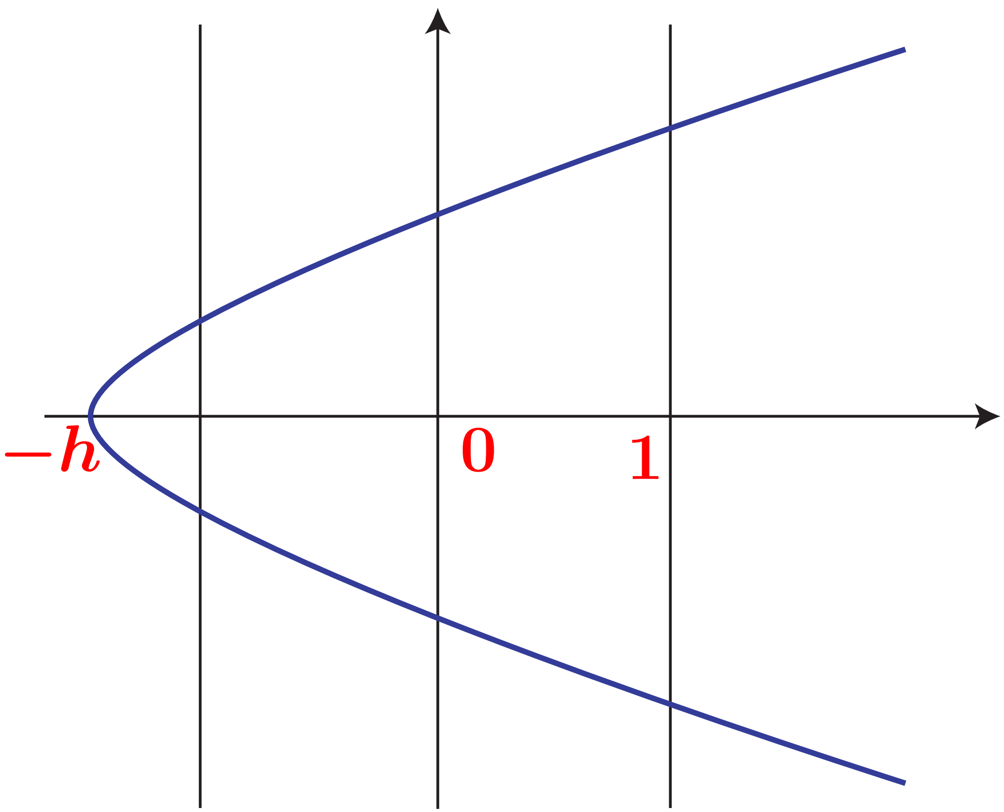

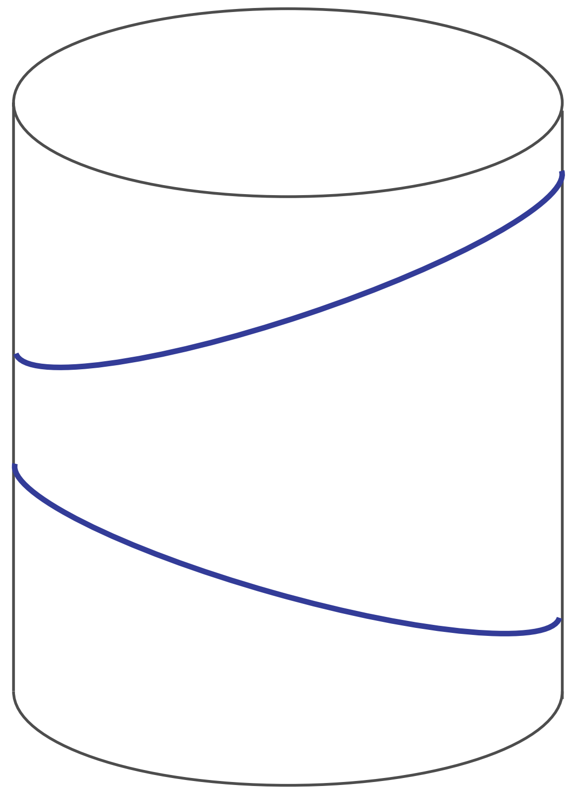

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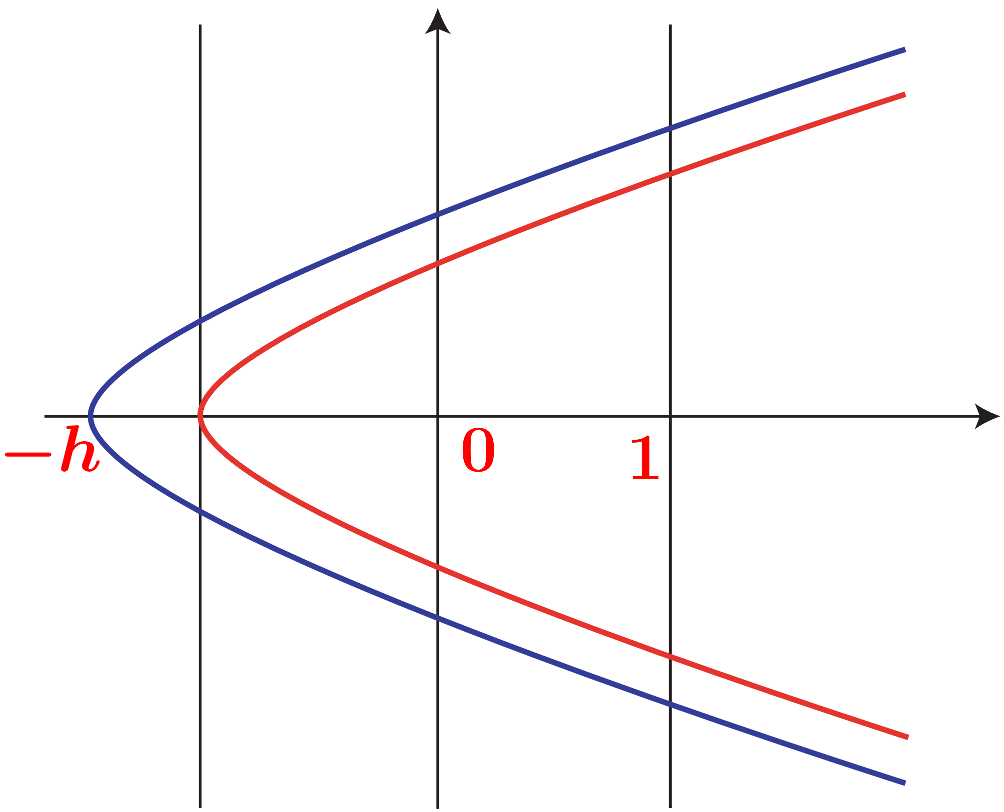

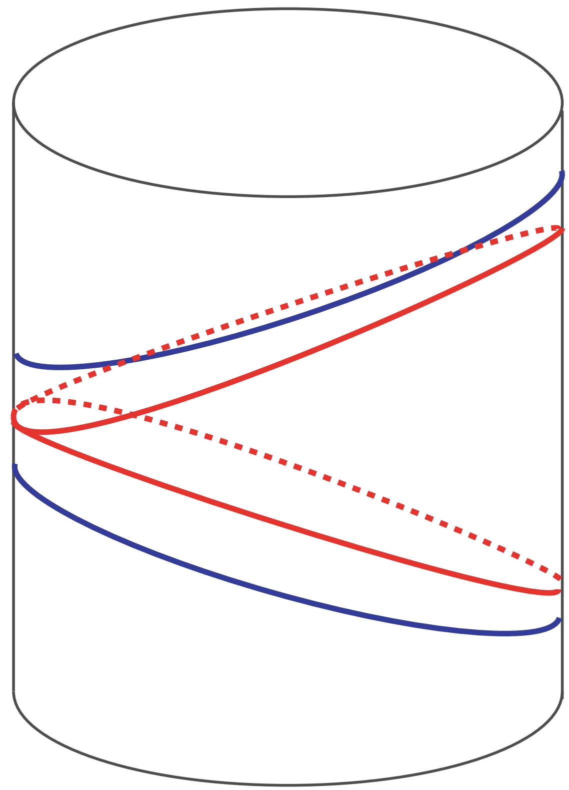

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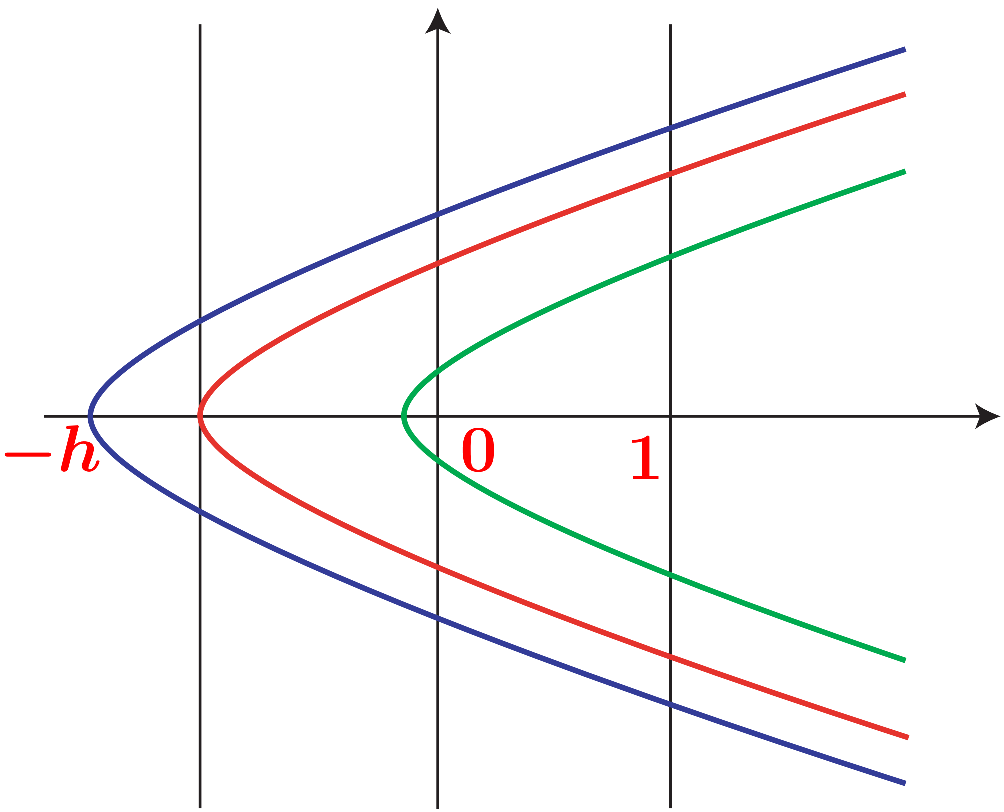

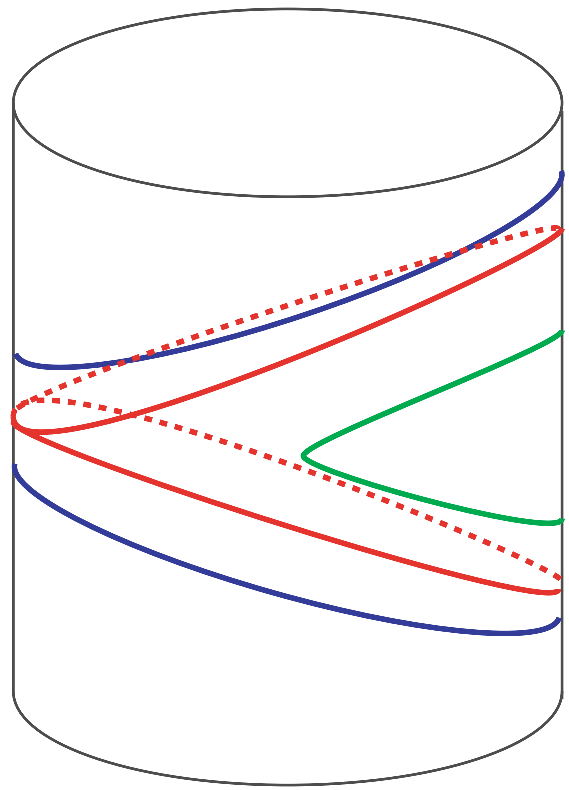

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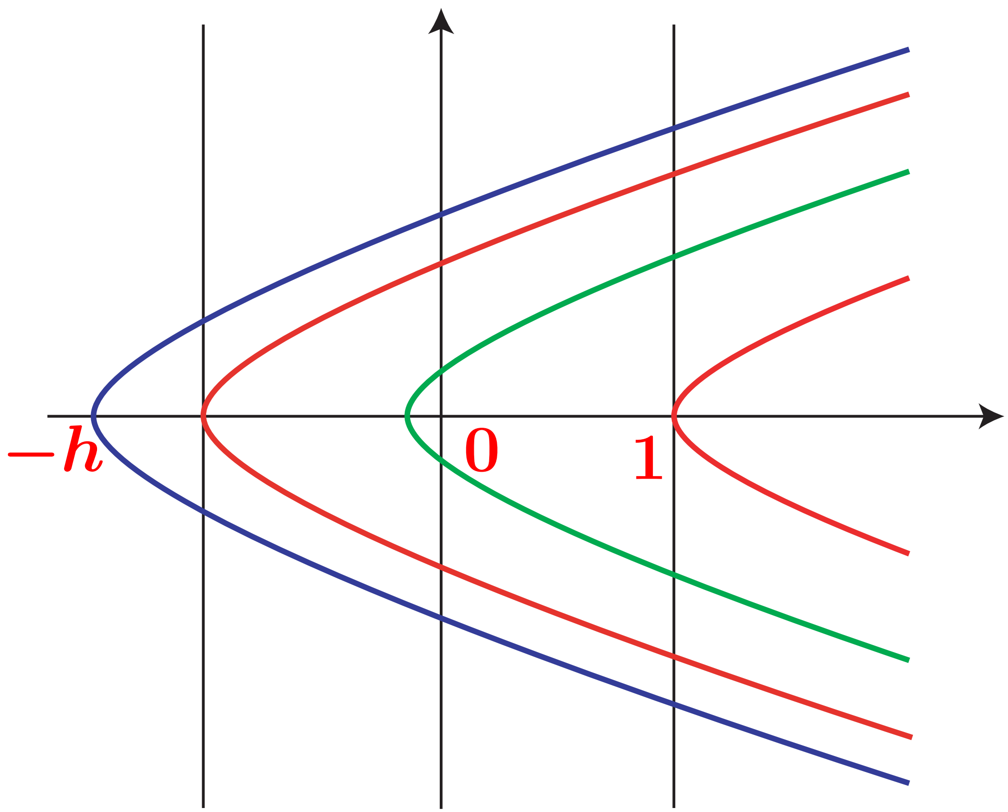

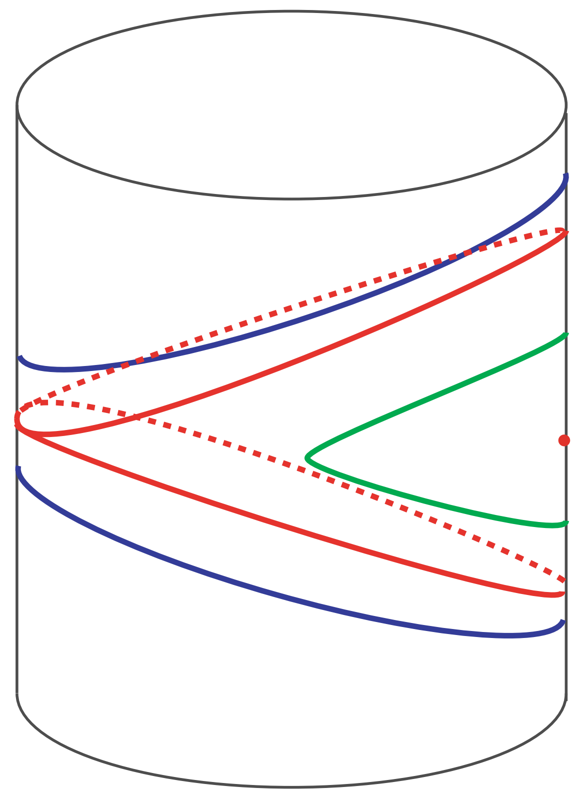

The linearized equation

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where $y = \delta\gamma''$, r and γ are solutions of the original differential system, namely $r(t)$ and $\gamma(t)$ are a (time) parametrization of the elliptic curve E_h.

Monodromy around the poles

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Haine's criterion (trivial monodromy) $\Rightarrow \frac{4}{A} \in \mathbf{Z}$.

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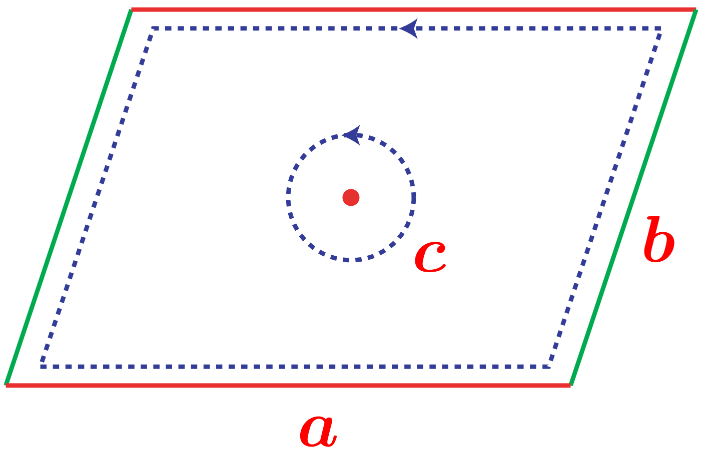

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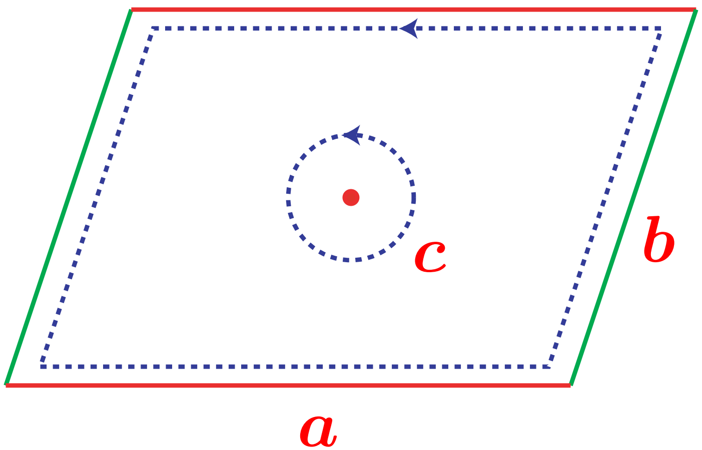

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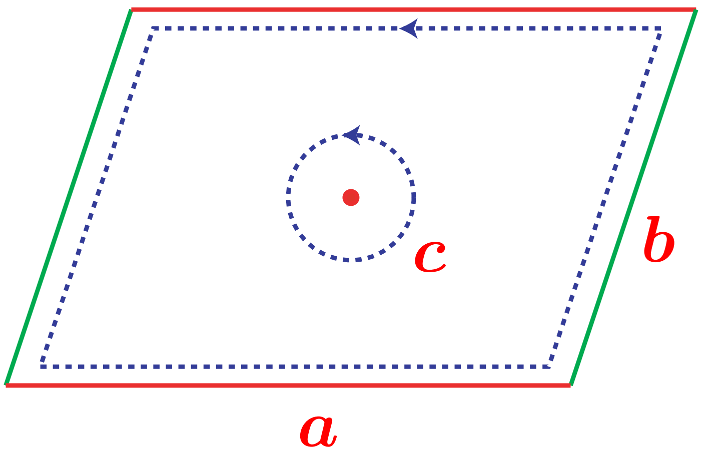

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Hence the monodromy group is not Abelian.

The Galoisian criterion

$$\begin{array}{ccccc} \text{(K)} & \Longrightarrow & \text{(H)} & \Longleftrightarrow & \mathrm{Gal}\ \text{abelian} \\ & & & & \Downarrow \\ \text{(L)} & \Longrightarrow & \text{(MR)} & = & \mathrm{Gal}^{\circ}\ \text{abelian} \end{array}$$

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As we are not sure that a and b are in the same component, $c = aba^{-1}b^{-1} \neq 1$ is not enough.

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Hence a subgroup conjugated with

$$H = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ with } \lambda \in \mathbb{C}^{\star} \right\}.$$

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$$G = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{C}^\star \right\} \cup \left\{ \begin{pmatrix} 0 & \mu^{-1} \\ -\mu & 0 \end{pmatrix} \mid \mu \in \mathbb{C}^\star \right\}$$

Our commutator $c \sim \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ will then be $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-2} \end{pmatrix}$, for some $\lambda \in \mathrm{R}^\star$, $\lambda \neq \pm 1$, Our commutator $c \sim \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ will then be $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-2} \end{pmatrix}$, for some $\lambda \in \mathrm{R}^\star$, $\lambda \neq \pm 1$,

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